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Editor

INTRODUCTION TO COMPLEX NUMBERS

Louis E. Diamond

Foreword.

The purpose of this article is to present an elementary introduction to complex numbers with a glance at some applications. The intent is to present the concepts intuitively, more or less as they probably originated, rather than formally by a rigorous procedure of definitions and postulates. Inevitably this leads to some repetitions in statements but in an elementary presentation this is not a pedagogical fault. A brief review of the history of complex numbers is interesting and relevant. The first introduction of imaginary numbers occurred in connection with the solution of cubic equations. During the 16th century Italian mathematicians of the then famous university of Bologna discovered that sometimes correct answers could be obtained more expeditiously if they assumed 1) a symbol, i, 2) $i^2 = -1$, and 3) in other respects treated i as an ordinary number. One of Cardano's problems was "Divide 10 into two parts whose product is 40." Cardano was intrigued by the answer but at the same time he was very doubtful of the method. For several hundred years mathematicians continued to play with the concept. Euler (1707-1783) achieved brilliant results by the use of complex numbers but the fundamental principles of their logic was either not deemed important or was completely misunderstood. Struik ways that in the 18th century "There was experimentation with infinite series, with infinite products, ... With the use of symbols such as 0, ∞, $\sqrt{-1}$, ... Much of the work of the leading mathematicians impresses us as wildly enthusiastic experimentation."

Wessel, a Danish mathematician, probably furnished the first logical foundation for complex numbers, but his paper "*On the Analytical Representation of Direction*," presented in 1797, was unknown to mathematicians until republished one hundred years later. Argand, a Paris accountant, also presented in 1806 a logical foundation but his work appears to have received very little attention. Actually Gauss (1777-1855) first used the phrase *complex number*, the symbol i for the imaginary unit, and introduced mathematicians to the true theory of these numbers. Gauss' work was followed by that of Cauchy (1789-1857) and that of Riemann (1826-1866). From their works has arisen the complex function theory which is basic and indispensable for advanced mathematics, and even for a full comprehension of the more elementary theorems of algebraic analysis. The point to be emphasized is this. More than 200 years passed before mathematicians placed complex numbers upon a firm logical basis.

The Equation $(1 + i\sqrt{3})^3 = -8$.

The study of imaginaries arose from the solution of cubic equations. Let $x^3 + 8 = 0$. Then $(x + 2)(x^2 - 2x + 4) = 0$. $(x+2)[(x-1)^2 + (\sqrt{3})^2] = 0$. Hence $x = -2$, or $(x-1)^2 = -(\sqrt{3})$. The quantity $(\sqrt{3})^2$ is a positive real number, and $-(\sqrt{3})^2$ is a negative real number. It is clearly brought out in the elementary study of exponents that the even root of a negative number cannot be either a positive number, a negative number, or zero. Hence the equation $x^3 + 8 = 0$ has only one solution, -2, in real numbers, and the equation $x^2 - 2x + 4 = 0$ has no solution in real numbers. In elementary work it is usual procedure at this point to introduce an "imaginary number," symbolized by i, whose square, i^2, is equal to -1, and which combines with other numbers according to the ordinary laws of algebra. Actually the student is passing from the algebra of real numbers to the algebra of complex numbers, and all the theorems of one system are not valid in the other.

By the device of changing -1 into a square, $(x-1)^2 = -(\sqrt{3})^2$ can now be rewritten as $(x-1)^2 = i^2(\sqrt{3})^2$ and hence $x - 1 = \pm i\sqrt{3}$, and $x = 1 \pm i\sqrt{3}$. Selecting the value $1 + i\sqrt{3}$, using the binomical expansion $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$, and assuming that i can be operated with as a real number, $(1 + i\sqrt{3})^3 = 1 + 3i\sqrt{3} + 9i^2 + (i^2)(i)(3\sqrt{3})$. Since $i^2 = -1$, this simplifies to -8 and hence $(1 + i\sqrt{3})$ is a cube root of -8. Similarly $(1 - i\sqrt{3})$ is shown to be a cube root of -8. A real number can have only one cube root. The operation with i has introduced two additional cube roots for -8. Evidently the algebra is no longer that of real numbers and we should not expect all real number theorems to be valid. In real numbers if $A^3 = B^3$, $A = B$. $(1 + i\sqrt{3})^3 = (1 + i\sqrt{3})^3 = (-2)^3$. $1 + i\sqrt{3} = -2$ or $i = -\sqrt{3}$ and $i^2 = \sqrt{3}$ which is clearly inconsistent with the assumption that $i^2 = -1$. In fact $(1 - i\sqrt{3})^3 = (1 + i\sqrt{3})^3 = (-2)^3$ but $1 - i\sqrt{3} \neq 1 + i\sqrt{3} \neq -2$.

The algebraic sum of $1 - i\sqrt{3}$ and $1 + i\sqrt{3}$ is 2. Multiplying $1 - i\sqrt{3}$ by $1 + i\sqrt{3}$ we obtain $1 + i\sqrt{3} - i\sqrt{3} - 3i^2$, or 4. The student might conclude with Cardano that he has divided two into two parts whose product is 4, that $1 - i\sqrt{3}$ is one of thse parts. Like Cardano he might conclude that this result was subtle, probably useless, and above all worthy of the name "imaginary."

Thses questions now arise. Can 'numbers' such as $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$ be factors of 4 in the sense that 1 and 4, and 2 and 2, are factors of 4? Can the sum of these two numbers be the real number two, in the sense that $1 + 1 = 2$? What is the geometrical interpretation of the numbers which have the form $A + Bi$, where A and B are taken from the real number field? we now consider the last question.

The Argand Diagram

A Swiss engineer, Argand, (1768-1822), utilized a simple geometric theorem in his interpretation of the symbol i. In Fig. 1 triangle ABC is right angled at B. OB, of length F, is perpendicular to the hypotenuse of ABC, dividing the hypotenuse into two segments, AO, whose length is E, and OC whose length is D. By the pythagorean theorem

$$(E + D)^2 = AC^2 = AB^2 + BC^2. \quad AB^2 = F^2 + E^2. \quad BC^2 = F^2 + D^2$$

$$ED = F^2 \quad \text{or} \quad E/F = F/D, \quad F = \sqrt{ED}.$$

If three numbers, E, F, and D have this relation, F is said to be the geometric mean of E and D.

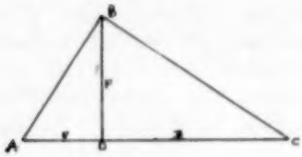

Fig. 1

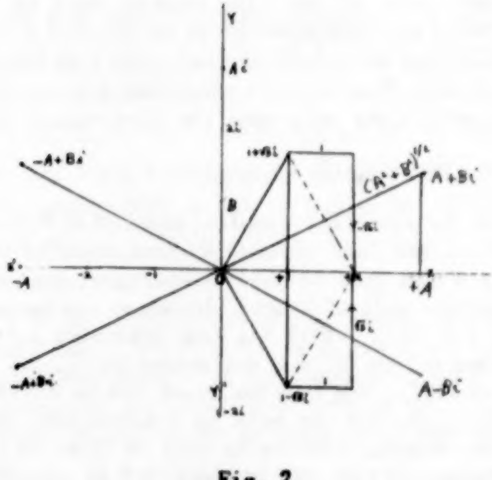

Fig. 2

In Figure 2 $x'x$ may be called the axis of reals. To the right of the origin 0, the real number +1 corresponds to the endpoint of an arbitrary unit of length. The number $+A$ corresponds to the endpoint

of an arbitrary length of A units to the right of 0. The numbers -1 and $-A$ correspond respectively to the endpoints of the unit of length, and of an arbitrary length of A units, to the left of 0. $Y'Y$, the axis of imaginaries, is perpendicular at 0 to the axis of reals. The triangle whose vertices are the points -1, +1, and B, is clearly right angled at B so that from the preceding theorem $F = \sqrt{(-1)(+1)} = \sqrt{-1} = i$. Analogously the geometric mean of $+A$ and $-A$ is $'Ai$. $\sqrt{(-A)(A)} = \sqrt{-A} = Ai$.

Argand considered that the "multiplication" of A by i was the algebraic equivalent of a geometric counterclockwise rotation about 0 of the directed line segment $0A$ through an angle whose measure was $\pi/2$ radians. The terminal point of the rotated segment became Ai, the length of the segment being unaltered. In other words i acted upon the directed magnitude $0A$ to change its direction by 90° counter-clockwise without changing its magnitude. A second "multiplication" by i rotates the line segment Ai counterclockwise about 0 through $\pi/2$ radians so that its terminal point is now $-A$. These two successive operations by i constitute the operation i^2 which is equivalent to the ordinary multiplication of A by -1. The operation i^3 brings the terminal point to $-Ai$, or $i^3 = i^2 i = -1i = -i$. The operation i^3 is equivalent to the operation $-i$. By defining the operation $-i$ as a clockwise rotation about 0 through $\pi/2$ radians, the operation $i^3 = -i$ is consistent. Four successive operations by i, or the operation i^4, brings the terminal point of the line segment back to $+A$. This is equivalent to ordinary multiplication of $+A$ by +1. $i^2 i^2 = (-1)(-1) = +1$. Complex numbers are thus conceived of, not simply as magnitudes, but as directed magnitudes. Particularly note that i as an operator has no effect upon magnitude but only upon the direction of the magnitude.

The Complex Number, $A + Bi$.

We first set up the so-called complex plane. Let 0 be an arbitrary fixed origin, $0x$ a fixed line through 0, analogous to the positive x axis, and $0x'$ a fixed line through 0 analogous to the negative x axis. With an arbitrary unit of length, distances are marked off on $0x$ and designated by 1,2,3,... . With the same arbitrary unit of length, distances are marked off on $0x'$ and designated by -1,-2,... . $x0x'$ is called the axis of reals. The complex plane can be divided into four quadrants by a line, called the axis of imaginaries, perpendicular to $x'0x$ through the origin. Essentially this is shown in Figure 2.

The complex number, $\pm A \pm Bi$ can be regarded as instructions for locating the position of its terminal point in the complex plane with respect to the chosen origin. This point is reached by two displacements, the first one along $x'0x$ to the number A on $0x$, if A is positive, to the number $-A$ on $0x'$ if A is negative. The symbol i now

rotates the displacement about A or $-A$, counterclockwise through $\pi/2$ radians if Bi is positive, clockwise if Bi is negative. The second displacement then proceeds a length B along this line perpendicular to $x'0x$ and parallel to the axis of imaginaries. Hence A Bi will be in the first quadrant with $-A - Bi$, in the third quadrant, both symmetrically placed with respect to the origin. $-A + Bi$ will be in the second quadrant with $A - Bi$ in the fourth quadrant, both symmetric with respect to the origin. The two directed displacements, A and Bi, determine a distance from the origin, usually symbolized by R, and commonly called the modulus or absolute value of the complex number. This distance has the magiitude $(A^2 + B^2)^{\frac{1}{2}}$. Two complex numbers, which differ only in the sign of the Bi term are called conjugate. The complex number is usually symbolized by z, its conjugate by $\bar{z}$, (bar above). As examples the terminal points of the conjugates, $1 + \sqrt{3i}$ and $1 - \sqrt{3i}$ are shown in Figure 2. Since the positive square root of $1^2 + (\sqrt{3})^2$ is 2, the modulus of each number is 2 and both terminal points lie on the circumference of a circle whose center is the origin and whose radius is 2 units in lenth. The third cube root of -8, -2, also lies on this circumference, all three roots spaced equally.

The Trigonometric Form of z.

The direction of the complex number $A + Bi$ is given both by $\phi = \sin^{-1} B/R$ and $\phi = \cos^{-1} A/R$. For example $1 + \sqrt{3}i$. $\phi = \sin^{-1} \sqrt{3}/2$. $\phi = \cos^{-1} 1/2$. $\phi = 60°$. $1 - \sqrt{3}i$. $\phi = \sin^{-1} -\sqrt{3}/2$. $\phi = \cos^{-1} 1/2$. $\phi = -60°$. The equation $\phi = \tan^{-1} B/A$ does not unambiguously give the quadrant in which the terminal point of the complex number lies. For example, for $1 + \sqrt{3}i$, $\phi = \tan^{-1} \sqrt{3}$ gives both the correct answer, $60°$, and the answer $240°$ which is valid for $-1 - \sqrt{3}i$. The measure in radians of the angle ϕ is called the argument or the amplitude of the complex number.

Since $A = R \cos \phi$ and $B = R \sin \phi$, $A + Bi$ has the equivalent representation $R \cos \phi + i R \sin \phi$ or $R(\cos \phi + i \sin \phi)$. ϕ is the numerically smallest angle that is consistent with the equations. The sine and cosine are periodic, with period 2π. This form of z, the complex number, is called the trigonometric form while $A + Bi$ is the normal form.

Addition of Complex Numbers

The "sum" of $A_1 + B_1 i$ and $A_2 + B_2 i$ is defined as follows. Carry out the operation which brings a point from the origin to the point

$A_1 + B_1 i$. Then follow this operation from the point $A_1 + B_1 i$ by a displacement which is parallel to and equal to the displacement which carries a point from the origin to $A_2 + B_2 i$. If $z_1 = A_1 + B_1 i$ and $z_2 = A_2 + B_2 i$, $z_1 + z_2 = z_3 = (A_1 + A_2) + (B_1 + B_2)i$. If points z_1 and z_2 are not collinear with point 0, the point z_3 is the 4th vertex of a parallelogram whose sides are $0z_1$ and $0z_2$. The addition of $1 - \sqrt{3i}$ and $1 + \sqrt{3i}$ is shown in Figure 2

$A + Bi$ and $-A - Bi$ represent two directed magnitudes whose moduli are equal but which are oppositely directed. Their sum is the complex number $(A - A) + (B - B)i = 0 + 0i$. This complex number is analogous to the real number zero.

The word *sum* in complex numbers therefore merely implies that we consider the combined effect of two displacements, i.e. the single displacement equivalent to the two displacements combined. An operation is called *associative* if, when it is used on three or more arbitrary elements, the result is independent of the manner in which the elements are grouped for computing. Symbolically $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$. It can be proved that addition is associative.

An operation is called *commutative* if the result obtained by its use on two arbitrary elements taken in a certain order is the same as the result obtained by its use on the same two elements taken in the opposite order. Addition is commutative. In Fig. 2 we intuitively see that $(1 + i\sqrt{3}) + (1 - i\sqrt{3}) = (1 - i\sqrt{3}) + (1 + i\sqrt{3})$.

Multiplication of Complex Numbers.

The product of $(A_1 + B_1 i)$ by $(A_2 + B_2 i)$ is defined as $(A_1 A_2 - B_1 B_2) + i(A_1 B_2 + A_2 B_1)$. The rules of real number multiplication are used, i.e. associative, commutative, and distributive with respect to addition. When we state that multiplication is distributive with respect to addition and subtraction, we mean that it is true for all elements z of the complex number system that $z_1(z_2 + z_3 - z_4) = z_1 z_2 + z_1 z_3 - z_1 z_4$ for example $(1 + i^2)(A + Bi) = 1(A + Bi) + i^2(A + Bi) = A + Bi - A - Bi = 0 + 0i$. The outstanding difference between real number multiplication and complex number multiplication is the change of 1^2 to -1 whenever it occurs. The geometrical interpretation of multiplication is more readily visualized when complex numbers are taken in their equivalent trigonometric form.

$$[R_1(\cos\phi_1 + i\sin\phi_1)][R_2(\cos\phi_2 + i\sin\phi_2) =$$

$$R_1 R_2[(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) + i(\cos\phi_1\sin\phi_2 + \cos\phi_2\sin\phi_1)]$$

$$= R_1 R_2[(\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)]$$

Clearly the modulus of the product is the product of the moduli of the factors, and the amplitude or argument of the product is the sum of the amplitudes of the factors, i.e. lengths are multiplied and angles are added. This rule can be extended to any number of factors. For example the product of N factors whose moduli are each unity is $\cos(\phi_1 + \phi_2 + \ldots + \phi_n) + i \sin(\phi_1 + \phi_2 + \ldots + \phi_n)$. If $\phi_1 = \phi_2 = \ldots = \phi_n$, an extremely important identity is obtained. $(\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi$, where n is a positive integer. The extension to n other than a positive integral exponent will be discussed later.

Operators

In visualizing geometrically the multiplication of complex numbers it is quite advantageous to regard the multiplier as an operator. For example the complex number i, acting upon a directed magnitude, leaves its magnitude unaltered but rotates its direction counterclockwise through an angle whose radian measure is $\pi/2$. The operation i^2 is the operation i carried out twice successively. This counterclockwise rotation of a directed magnitude through an angle of $\pi/2 + \pi/2 = \pi$ radians is equivalent to multiplication by -1. The operation i^4 is equivalent either to the operation i or the operation $-i$ repeated consecutively four times. A rotation through 2π radians either clockwise or counterclockwise leaves the direction unaltered and is equivalent to multiplication by +1. Multipliers such as 2, 3, 4, ... do not alter the direction of the directed magnitude but they do multiply its magnitude by 2, 3, 4, ... Multipliers such as -2, -3, -4, ... reverse the direction of the directed magnitude and multiply its magnitude by 2, 3, 4, ...

If z is a complex number, and $z^2 = i$, we visualize z as an operator which leaves magnitude unchanged but rotates a directed magnitude counterclockwise through an angle whose measure is $\pi/4$ radians. This follows directly from the fact that in multiplication angles are added, and the operator $\sqrt{i}$ acting twice successively rotates the directed magnitude through $\pi/4 + \pi/4 = \pi/2$ radians. The visualization of $z^2 = -i$ follows directly from the fact that $-i$ represents a clockwise rotation through $\pi/2$ radians.

The complex number $A + Bi$ can also be considered an operator or direction coefficient which rotates any directed magnitude through an angle theta which is defined by its sine and cosine as a function of A and B. It also multiplies the modulus of the directed magnitude by $(A^2 + B^2)^{\frac{1}{2}}$. For example $1 + \sqrt{3}\, i$ rotates any directed magnitude counterclockwise through an angle of $\pi/3$ radians and multiplies its length by 2. When $1 + \sqrt{3}i$ operates upon itself, the resulting

complex number has a modulus of $(2)(2) = 4$, and $\phi = \pi/3 + \pi/3 = 2\pi/3$. An operation on this product by $1 + \sqrt{3}\, i$ yields a complex number whose modulus is $(4)(2) = 8$, and whose amplitude is π.

At this point we might ask, "What connection has an operator with a number?" For an indirect answer consider the speedometer in an auto. The rotation of the wheel is an operation, and a certain operation adds one to the mileage. An adding machine acts on a similar principle. From a practical standpoint any operation consisting of a change in magnitude and of a rotation can be set up in the form of a complex number, $A + Bi$, since it controls two measures simultaneously and covers a plane instead of a line. As an example consider a copper coil rotating in a magnetic field whose direction and magnitude are changing with time.

In a similar manner we can regard the factor $(\cos\phi + i \sin\phi)$ as an operator or direction coefficient which rotates any directed magnitude counterclockwise through an angle of theta radians and which leaves the magnitude unchanged. The factor $R(\cos\phi + i\sin\phi)$ acts in exactly the same manner but multiplies the magnitude by R. The operator $1(\cos\pi/2 + i\sin\pi/2)$ completely fulfills the earlier definition of i.

The operator $[\cos(-\phi) + i\sin(-\phi)]$ rotates a complex number through an angle $-\phi$. The cosine is an even function, i.e. $\cos(-\phi) = \cos\phi$. The sine is an odd function, i.e. $\sin(-\phi) = -\sin\phi$. Hence we can equivalently write $(\cos\phi - i\sin\phi)$. If a complex number is operated upon by $(\cos\phi + i\sin\phi)$ and then by $(\cos\phi - i\sin\phi)$ the rotation through ϕ and then through $-\phi$ with magnitude unchanged leaves the complex number unchanged. The operator product $(\cos\phi + i\sin\phi)(\cos\phi - i\sin\phi) = \cos^2\phi + \sin^2\phi = 1$, i.e. it leaves magnitude and direction unaltered. By analogy with real numbers $(\cos\phi + i\sin\phi)(\cos\phi + i\sin\phi)^{-1} = (\cos\phi + i\sin\phi)^0 = 1$. Hence $(\cos\phi + i\sin\phi)^{-1}$ is defined as $(\cos\phi - i\sin\phi)$. A meaning is thus given to negative exponents.

Division

Since division is the inverse of multiplication, z_1/z_2 is that complex number which yields z_1 when multiplied by z_2.

$$\begin{aligned} z_1/z_2 &= R_1(\cos\phi_1 + i\sin\phi_1)/R_2(\cos\phi_2 + i\sin\phi_2) \\ &= (R_1/R_2)[(\cos\phi_1 + i\sin\phi_1)(\cos\phi_2 + i\sin\phi_2)^{-1}] \\ &= (R_1/R_2)[\cos(\phi_1 - \phi_2) + i\sin(\phi_1 - \phi_2)]. \end{aligned}$$

The modulus of the quotient is equal to the quotient of the moduli and the amplitude to the difference of amplitudes of the dividend

and divisor, i.e. in division we divide moduli and take the difference of angles.

As in real numbers z_2 must not be the complex number zero. Division by zero has no meaning. If $(C + Di) \neq 0 + 0i$, then

$$(A + Bi)/(C + Di) = (A + Bi)(C - Di)/(C + Di)(C - Di)$$

$$= (AC + BD)/(C^2 + D^2) - i[(BC - AD)/(C^2 + D^2)]$$

Since

$$(\cos\phi + i\sin\phi)^{-n} = [\cos(-\phi) + i\sin(-\phi)]^n = \cos(-n\phi) + i\sin(-n\phi)$$

the identity developed previously holds for negative integral exponents.

Equality, real part, imaginary part.

If $A + Bi = C + Di$, then by definition $A = C$, and $B = D$. From geometrical considerations this definition offers no contradictions. Deeply established in the literature are the words real part for A, and imaginary part for B of the complex number $A + Bi$. These names, real and imaginary, originated in the days before there was a logical basis for complex numbers. The axis which we have called $x'0x$ is called the axis of reals and the axis through the origin perpendicular to the axis of reals is called the axis of imaginaries. Despite their inappropriateness in the modern approach, their usage is so firmly established in the literature, and convenience of expression is so great that their usage cannot be avoided. The use of the words real number; imaginary number, pure imaginary, rational integer, etc. merely implies that these numbers are to be regarded as imbedded in the complex number system as subsets thereof. In a succeeding paragraph we shall bring up certain distinctions between real numbers and complex numbers.

If two complex numbers are equal, it follows from the definition of equality that their moduli are equal. Observe, however, that the converse is not true. Geometrically if the moduli of two complex numbers are equal, their terminal points lie on the circumference of a circle whose center is the origin and whose radius has the length of the modulus. Since two conjugate complex numbers differ only in the sign of i, their moduli are the images of each otherin the real axis. (see Figure 2)

The Symbolic form of the complex number.

Since a complex number consists of a magnitude and the radian measure of an angle, this number pair offers a convenient geometric representation of the complex number. This number pair is ordered,

i.e. it is agreed to write the modulus first and the radian measure of the angle second. The modulus of a complex number is always positive and is usually designated by R. The radian measure of the angle is designated by ϕ, hence the general number pair is (R, ϕ). Clearly the same geometric position is reached if ϕ is replaced by $\phi + 2\pi$, $\phi + 4\pi$, ... The result is ambiguous unless it is agreed that

$$0 \leq \phi < 2\pi \qquad \text{or} \qquad 0 \geq \phi > -2\pi$$

As examples $(1,0)$ is equivalent to $1 + 0i$, $(1,\pi)$ to $-1 + 0i$, $(1,\pi/2)$ to $0 + i$. We can also write $i^4 = (1,0)$,, $.-i = (1,3\pi/2) = (1, -\pi/2)$. If $z_1 = (R_1,\phi_1)$ and $z_2 = (R_2,\phi_2)$, $z_1 z_2$ is defined as $(R_1 R_2, \phi_1 + \phi_2)$. z_1/z_2, z_2 not the complex number zero, is defined as $(R_1/R_2, \phi_1 - \phi_2)$.

With this notation consider an operator ω whose modulus is 1 and whose radian measure is $2\pi/3$, i.e. the complex number $-\frac{1}{2} + i\sqrt{3}/2$. Operating upon itself we obtain a complex number ω^2, $(1,4\pi/3)$. Let us now operate upon ω^2 by itself. The product, ω^4, is $(1,8\pi/3)$. This is the same as $(1,2\pi/3)$, or ω. ω, not equal to ω^2, is a square root of ω^2, and yet when we square ω^2, we find the square equal to ω, the square root of ω^2. Such an interesting result leads naturally to an investigation of the operation of evolution, which is very important in the theory of complex numbers.

Square roots, postulates of sign and order.

Let (R_1,ϕ_1) and (R_2,ϕ_2) be two ways of denoting the same complex number. By the definition of equality $R_1 = R_2$, since both are positive. $\phi_1 = \phi_2$, or ϕ_1 and ϕ_2 differ by an integral multiple of 2π. Let $z = (R,\phi)$. Then $u = (\rho,\omega)$ is said to be a square root of z if $u^2 = z$. In that case $(\rho^2,2\omega)$ is identical with (R,ϕ). Then $\rho = +\sqrt{R}$, $2\omega = \phi + 2n\pi$, $\omega = \phi/2 + n\pi$, n an integer (including zero). Hence ω has the two values, $\phi/2$ and $\phi/2 + \pi$, or angles that differ from them by an even integral multiple of π. Consequently $u_1 = (\sqrt{R}, \phi/2)$ and $u_2 = (\sqrt{R}, \theta/2 + \pi)$. In particular if $\phi = 0$ and $R = 1$, the two square roots become $(1,0)$ and $(1,\pi)$. The important point is this. Every complex number has two square roots. We have already seen that every complex number has three cube roots. Contrast this situation with that applicable to real numbers.

In this paragraph we confine ourselves exclusively to the field of real numbers. Every positive real number has two square roots, one positive, one negative. A negative real number has no square root. For example the real number, $+16$ has the two square roots, $+4$ and -4. However -4 has no square root. Again $+4$ has two square

roots, +2 and -2. Again -2 has no square roots. Loosely speaking, The square roots descend continuously on the positive side only. The convention is also adopted that $\sqrt{A}$ for a positive real number, A, means the positive square root of A. Every positive real number has two and only two nth roots, n an even integer. Every positive real number has one and only one nth root, n an odd integer. A negative number has no even nth roots and has one and only one nth root, n an odd integer, The law of trichotomy postulates that a real number is either positive, negative, or zero. Geometrically the real numbers correspond to points on a line. They are livearly ordered, i.e. if a, b, and c are arbitrary real numbers, $a \neq b$ implies $a > b$ or $b > a$. $a < b$ implies $a \ngtr b$, and $a < b$ and $b < c$ imply $a < c$.

Geometrically the complex numbers are represented by the points in a plane and they are not linearly ordered. If z and u are two arbitrary complex numbers, either $z = u$, or $z \neq u$. No non-zero point in the plane occupies a preferred position and there is no satisfactory way to define the magnitude of complex numbers so that they can be linearly ordered. Every complex number has a square root which is also a complex number. Hence the postulates of the law of trichotomy cannot be fulfilled.

Norms and absolute values.

The product of two conjugate numbers, z and $\bar{z}$, is called the norm of z, $n(z)$. $n(z) = n(a + bi) = (a + bi)(a - bi) = a^2 + b^2$. The absolute value, or modulus of z, denoted by $|z|$ is the non-negative square root of the norm. $|z| = \sqrt{a^2 + b^2}$. The absolute value of a complex number is always a real number.

There are two important theorems which are stated without proof. The norm of a product is equal to the product of the norms of its factors. Symbolically if

$$z_1 = a + bi, \quad z_2 = c + di, \quad n(z_1 z_2) = n(z_1)n(z_2) = (a^2 + b^2)(c^2 + d^2).$$

The absolute value of a sum of complex numbers is less than or equal to the sum of the absolute values.

$$|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$$

The symbol i in Multiplication.

Let us examine by simple examples the reasons why the symbol i is useful in multiplying two complex numbers. By definition the product of $(a + bi)$ by $(a + bi)$ has the modulus $a^2 + b^2$. By actual algebraic multiplication with the usual convention about i and i^2, $(a + bi)(a + bi) = (a^2 - b^2) + 2abi$. The modulus of this product is the square root of $[(a^2 - b^2)^2 + (2ab)^2]$, which is $a^2 + b^2$.

as required by the definition.

In multiplication the angles are added. Let the argument of $a+bi$ be ϕ whose cosine is $a/(a^2+b^2)^{\frac{1}{2}}$ and whose sine is $b/(a^2+b^2)^{\frac{1}{2}}$. Then the argument of $(a+bi)^2$ must be 2ϕ. Entirely independent of complex numbers it can be proved that $\cos 2\phi = \cos^2\phi - \sin^2\phi$ and $\sin 2\phi = 2\cos\phi\sin\phi$. The complex number $(a^2-b^2)+2abi$ has an argument whose cosine is $(a^2-b^2)/(a^2+b^2)$. But this is $\cos^2\phi - \sin^2\phi$ as required. The sine of the argument is $2ab/(a^2+b^2)$ which is sine 2ϕ. The use of the symbol i answers the purpose exactly.

Similarly algebraic multiplication shows that $(a+bi)^3 = (a^3-3ab^2)+i(3a^2b-b^3)$. The modulus of this complex number is

$$[(a^3-3ab^2)^2 + (3a^2b-b^3)^2]^{\frac{1}{2}}$$

$$= (a^6-6a^4b^2+9a^2b^4+9a^4b^2-6a^2b^4+b^6)^{\frac{1}{2}} = [(a^2+b^2)^3]^{\frac{1}{2}}$$

exactly as required by the definition of multiplication. The cosine of the argument of this complex number is

$$(a^3-3ab^2)/(a^2+b^2)^{3/2} =$$

$$4\left[\frac{a}{(a^2+b^2)^{\frac{1}{2}}}\right]^3 - 3\left[\frac{a}{(a^2+b^2)^{\frac{1}{2}}}\right] = 4\cos\phi - 3\cos\phi.$$

Independently of complex numbers it can be proved that this is equivalent to $\cos 3\phi$, the argument demanded by the definition of multiplication. Similarly the sine of the argument is

$$(3a^2b-b^3)/(a^2+b^2)^{3/2}$$

which is equivalent to

$$3\sin\phi - 4\sin^3\phi,$$

and this in turn is equivalent to sine 3ϕ.

Applications to trigonometric formulas.

From the definition of division

$$(\cos\omega + i\sin\omega)(\cos\phi - i\sin\phi) = \cos(\omega-\phi) + i\sin(\omega-\phi).$$

By multiplications we have

$$(\cos\omega\cos\phi + \sin\omega\sin\phi) - i(\cos\omega\sin\phi - \sin\omega\cos\phi) =$$

$$\cos(\omega-\phi) + i\sin(\omega-\phi).$$

From the definition of equality of complex numbers

$$\cos(\omega - \phi) = \cos\omega \cos\phi + \sin\omega \sin\phi.$$

$$\sin(\omega - \phi) = \cos\omega \sin\phi - \sin\omega \cos\phi.$$

Set $\phi = -\phi$.

$$\cos(\omega + \phi) = \cos\omega \cos\phi - \sin\omega \sin\phi.$$

Set $\omega = \phi$.

$$\cos 2\phi = \cos^2\phi - \sin^2\phi.$$

Since $\sin^2\phi = 1 - \cos^2\phi$,

$$\cos 2\phi = 2\cos^2\phi - 1 \text{ and } \cos 2\phi = 1 - 2\sin^2\phi.$$

Notice the ease with which trigonometric formulas can be obtained!

Application to uniform angular motion.

Since i rotates a directed magnitude through $\pi/2$ radians counterclockwise, we should logically expect that $(-1)^{1/3} = (i^2)^{1/3} = i^{2/3}$ should rotate a directed magnitude through $\pi/3$ radians counterclockwise. $(-1)^{1/4} = (i^2)^{1/4} = i^{2/4}$ should rotate a directed magnitude through $\pi/4$ radians, and $i^{2/n} = (\cos \pi/n + i \sin \pi/n)$. If $n = \pi$, $i^{2/\pi}$ is the operator which rotates a directed magnitude counterclockwise through one radian. A directed magnitude with constant modulus is rotating with a constant angular velocity, $\omega = 2\pi f$ radians per second. Time in seconds is measured from $T = 0$ when the directed magnitude lies positively directed along $0x$. At any subsequent time, T, the directed magnitude will have traversed ωT radians

$$(i^{2/\pi})^{\omega T} = (\cos\omega T + i \sin\omega T) = (i^{2/\pi})^{2\pi f T} = i^{4fT}.$$

Let $z = (\cos\omega T + i \sin\omega T)$. The right hand side of this equation is an operator which rotates a directed magnitude through ωT radians per second, counterclockwise. This is a fundamental equation for uniform angular motion. Its real part is the projection of the directed magnitude along $x'0x$, the axis of reals. Thus the real part corresponds exactly to the usual formulation of simple harmonic motion as the projection of a uniform circular motion on a diameter. $(\cos\omega T - i \sin\omega T)$ represents clockwise or negative rotation. When the rotation of z is given as a function of time the operator ϕ is thus changed to ωT.

Demoivre's Theorem.

A French mathematician, Abraham De Moivre (1667–1754), who left France after the revocation of the Edict of Nantes and settled in

London, has attached his name to an important theorem, given to-day in the form

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi.$$

n is an exponent, integral, rational or irrational, and either positive or negative. (The derivation for integral n, positive or negative, has already been given in preceding paragraphs).

This theorem is so thoroughly covered in texts that only one basic proposition depending upon it will be briefly considered. Let n be a rational integer.

$$r^{1/n}\left[\cos\frac{\phi + 2\pi k}{n} + i\sin\frac{\phi + 2\pi k}{n}\right],\quad k = 0,1,2,\ldots,(n-1).$$

By giving k the assigned values n complex numbers are obtained. If $k = n$ a complex number is obtained with the same amplitude as that obtained by setting $k = 0$. If $k = n + 1$ a complex number is obtained with the same amplitude as that obtained by setting $k = 1$. Since the modulus r is a length, it is a positive real number (or isomorphic with a real number) which therefore has one real nth root. The modulus is the same for all n complex numbers. The amplitudes of any two of these complex numbers differ by less than 2π, so that no two of them are equal. Hence each represents a different complex number. However the nth power of each one of them is the same complex number. $r(\cos\phi + i\sin\phi)$, since the values of ϕ are restricted. Owing to the periodicity of the trigonometric functions involved no new complex numbers can be obtained by giving to k values greater than $(n - 1)$. The conclusion is that any complex number whatsoever has n different nth roots. It can be proved that every complex number has n and only n nth roots.

This fact has some interesting consequences. For example we have already seen that if $z_1^3 = z_2^3$, z_1 is not necessarily equal to z_2. In complex numbers two unequal numbers may be the squares of each other, i.e. $z_1 \neq z_2$. $z_1 = z_2^2$. $z_1^2 = z_2$. For example

$$-1/2 + i\sqrt{3}/2 \neq -1/2 - i\sqrt{3}/2.$$

$$-1/2 + (\sqrt{3}\,i)/2 = (-1/2 - i\sqrt{3}/2)^2.$$

$$(-1/2 + i\sqrt{3}/2)^2 = -1/2 - i\sqrt{3}/2.$$

Of course this is a specially selected case but this is never possible in real numbers. And again

$$(\sqrt[n]{z_1})(\sqrt[n]{z_2}) \neq \sqrt{z_1 z_2}$$

for every z_1 and z_2.

Applications of De Moivre's Theorem.

There are theorems which assure us that all the real trigonometric identities hold for complex z.

Let $z = \cos\phi + i\sin\phi$. Then $z^{-1} = \cos\phi - i\sin\phi$.

By De Moivre's Theorem

$$z^n = \cos n\phi + i\sin n\phi. \qquad z^{-n} = \cos n\phi - i\sin n\phi.$$

$$z + z^{-1} = 2\cos\phi. \qquad z^n + z^{-n} = 2\cos n\phi.$$

Similarly $z^n - z^{-n} = 2i\sin n\phi$. It is desired to find an expression for $\cos^6\phi$ in terms of multiple angles of ϕ.

$$(z + z^{-1})^6 = (2\cos\phi)^6 = 2^6\cos^6\phi.$$

Expanding $(z + z^{-1})^6$ by the binomial theorem we have

$$z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$$

This is

$$(z^6 + z^{-6}) + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$$

This is

$$2\cos 6\phi + 12\cos 4\phi + 30\cos 2\phi + 20.$$

Equating this expression to $2^6\cos^6\phi$ we have

$$\cos^6\phi = \frac{\cos 6\phi + 6\cos 4\phi + 15\cos 2\phi + 10}{32}.$$

A theoretical application of Demoivre's Theorem is interesting. The equation $x^n - 1 = 0$ is called a binomial equation. Its combination with a reciprocal equation offers a very powerful tool for the solution of equations. We shall give in outline a specific example.

$$x^7 + 1 = 0. \qquad (x + 1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1) = 0.$$

From Demoivre's theorem we can obtain the seven 7th roots of -1, i.e. the complete solution of $x^7 + 1 = 0$. Since -1 is the solution of the factor $(x + 1)$ equated to zero, the other six roots are the roots of the second factor which, equated to zero, is a sextic. The six roots are $\cos\pi/7 \pm i\sin\pi/7$, $\cos 3\pi/7 \pm i\sin 3\pi/7$, $\cos 5\pi/7 \pm i\sin 5\pi/7$. The sextic equation can be equivalently written as a reciprocal equation.

$$(x^3 + 1/x^3) - (x^2 + 1/x^2) + (x + 1/x) - 1 = 0.$$

If we set $x = \cos\phi + i\sin\phi$, then $1/x = \cos\phi - i\sin\phi$, and (

$(x + 1/x) = 2 \cos \phi$, where ϕ has the values $\pi/7$, $3\pi/7$, and $5\pi/7$. Now if a is a real number greater than zero, and a given cubic can be written in the form

$$(ay)^3 - (ay)^2 - 2\,(ay) + 1 = 0,$$

Then by setting $x + 1/x = ay$, the above cubic becomes

$$(x^3 + 1/x^3) - (x^2 + 1/x^2) + (x + 1/x) - 1 = 0$$

Hence $ay = 2 \cos \phi$, where ϕ has the values $\pi/7$, $3\pi/7$, and $5\pi/7$. Similarly cubics of the form

$$(ay)^3 + (ay)^2 - 2(ay) - 1 = 0,$$

which is identical with the cubic of the preceding paragraph except for sign, corresponds to

$$(x^3 + 1/x^3) + (x^2 + 1/x^2) + (x + 1/x) + 1 = 0,$$

or the sextic

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$$

If

$$x + 1/x = 2 \cos \phi = ay, \ \phi \ = \ 2\pi/7, \ 4\pi/7, \ \text{and} \ 6\pi/7.$$

For example

$$27y^3 + 36y^2 - 96y - 64 = 0$$

is of the form

$$(ay)^3 + (ay)^2 - 2(ay) - 1 = 0,$$

where $3y/4 = 2 \cos \phi$. For the value of ϕ, $2\pi/7$, $2 \cos \phi = 1.247 = 3y/4$, and y is approximately 1.663.

Complex numbers as an ordered pair of real numbers.

Complex numbers have been considered more or less intuitively from a geometrical standpoint. The complex number (R, ϕ) is an ordered number pair since a radian measure is a number. However a radian is the ratio of an arc length to a radius. Now an arc length is not by any means a simple concept but is actually quite abstract. Technically the length of an arc is the upper bound of all chord sums. The fact that we agree to a single length guarantees the existence of an endless number of lengths. To divorce the complex number from all such geometrical considerations, an attitude of many mathematicians today, the complex number has been defined as an ordered pair, (a,b), of real numbers. By postulates and definitions of operations it can be shown that the technical requirements of a number system are fulfilled. Of course this becomes a much more abstract and rigorous presentation than that given.

$(a,b) + (c,d) = (a + c,\ b + d)$. $(a\ b) - (c,d) = (a - c,\ b - d)$

$(a,b)(c,d) = (ac - bd,\ ad + bc)$

1 becomes $(0,1)$ which has the property that $(0,1)(0,1) = (-1,0)$

$\sqrt{-1}$ has a meaning only when regarded in this light.

$(b,0)(0,1) = (0,b)$, i.e. $(0,b)$ is equivalent to bi

$(a,\ 0) + (0,b)$ is equivalent to $a + bi$.

Technically the complex numbers form a field. They have another property that is almost unique. The n roots of any polynomial equation of degree n, greater than one, with complex coefficients may be expressed as complex numbers, or technically, every polynomial of degree greater than one is reducible. The complex number system is algebraically closed and no extension of the complex number system is possible

Conclusion.

The applications shown have been largely confined to trigonometry. However it would be possible to give examples in geometry, the theory of numbers, the theory of equations etc. If multiplication is not involved complex numbers may be used for vectors. One of the most interesting and intriguing parts of complex number theory is tied up with the expansion of $\cos\phi$, $\sin\phi$, and e^x in convergent infinite series. In real numbers ϵ^x is a simple type of exponential function, somewhat like 2^x or 3^x, since ϵ is an irrational number lying between 2 and 3. However it has unusual properties which earn for it a special name, epsilon. We find that $r\epsilon^{i\theta}$ becomes a natural form of expression for the complex numbers. Problems of sines and cosines become problems about exponentials. We encounter expressions such as $\epsilon^{\pi i} = -1$ which has been called the most remarkable equation in mathematics, connecting as it does the numbers epsilon, π, i and -1. The reader will find that an investigation along these lines will be intriguing and stimulating.

Milford Texas.

An Infinite Descent Method to Prove Pythagorean Principle

J.M. Gandhi

*We know that the results

$$(1)\quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} \qquad (2)\quad \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$$

can be derived without the use of Pythagorean principle.
Squaring (1) and (2) and adding we get

$$(3)\quad \sin^2 A + \cos^2 A = (\cos^2\frac{A}{2} + \sin^2\frac{A}{2})$$

Similarly

$$(4)\quad \sin^2\frac{A}{2} + \cos^2\frac{A}{2} = (\sin^2\frac{A}{4} + \cos^2\frac{A}{4})$$

Hence

$$(5)\quad \sin^2 A + \cos^2 A = (\cos^2\frac{A}{2} + \sin^2\frac{A}{2})^2 = (\cos^2\frac{A}{4} + \sin^2\frac{A}{4})^4$$

$$= \ldots\ldots\ldots = (\cos^2\frac{A}{2K} + \sin^2\frac{A}{2K})^{2K} \ldots\ldots\ldots .$$

It can be proved that

$$(6)^{**}\quad \lim_{K\to\infty}(\cos^2\frac{A}{2K} + \sin^2\frac{A}{2K})^{2K} = 1$$

$$\therefore\ \sin^2 A + \cos^2 A = 1$$

Applying the definitions of sin A and cos A we shall get the required result.

* For reference see Plane Trigonometry by S.L.Loney, Art. 88.

** Here we have to assume

$$\frac{d}{d\theta}\sin\theta = \cos\theta,\quad \frac{d}{d\theta}\cos\theta = -\sin\theta,$$

and both can be derived using the results of Art. 88 of the above book and definition of differentiation.

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ELECTRON TRAJECTORIES IN A COMBINED ELECTRIC AND MAGNETIC FIELD

Chester R. Morris

This paper represents a portion of the work done by the author while employed as research mathematician by the National Union Radio Corporation, Electronic Research Division, Orange, New Jersey.
The work of Bradley at Philco Corporation in an unpublished paper is extended in this paper.

1. *Statement of Problem and Assumptions.*

We wish to find the path of a lone electron for the case of an electric potential field

$$\phi = \frac{\alpha}{2}\,(x^2 + y^2 - 2z^2), \tag{1}$$

combined with a uniform magnetic field $\mathbf{B}$ in the z-direction, where rectangular coordinates are used.

We use M.K.S. unites in this paper. Relativistic effects are ignored and we define the electron to be of constant mass. Also we disregard the size of the electron and define it to be a mathematical point with charge and mass.

2. *Derivation of Differential Equations.*

Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be unit vectors in the x, y and z directions respectively. Then the electric field is

$$\mathbf{E} = -\nabla\phi = -\,(\alpha x\mathbf{i} + \alpha y\mathbf{j} - 2\alpha z\mathbf{k}), \tag{2}$$

and the magnectic field is

$$\mathbf{B} = B\,\mathbf{k}, \tag{3}$$

where B denotes the scalar magnitude of $\mathbf{B}$ and is positive when $\mathbf{B}$ is in the direction of increasing z, negative when $\mathbf{B}$ is in the direction of decreasing z.

The electron is defined to move according to Newton's law of motion,

$$\mathbf{F} = m\frac{d^2\mathbf{R}}{dt^2} = m\ddot{\mathbf{R}}$$

where $\mathbf{R}$ denotes a vector from the origin to the point (x, y, z), m the mass of the electron and $\mathbf{F}$ the force on the electron. Also

$$\mathbf{F} = -\epsilon[\mathbf{E} + (\dot{\mathbf{R}} \times \mathbf{B})],$$

where ϵ denotes the absolute value of the charge of an electron. Thus

$$m\ddot{\mathbf{R}} = -\epsilon[\mathbf{E} + (\dot{\mathbf{R}} \times \mathbf{B})]. \tag{4}$$

Now

$$\mathbf{R} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z. \tag{5}$$

Differentiating

$$\dot{\mathbf{R}} = \mathbf{i}\dot{x} + \mathbf{j}\dot{y} + \mathbf{k}\dot{z}\ . \tag{6}$$

Differentiating again

$$\ddot{\mathbf{R}} = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z}\ . \tag{7}$$

Substituting (2), (3), (6) and (7) into (4),

$$m\ddot{x}\mathbf{i} + m\ddot{y}\mathbf{j} + m\ddot{z}\mathbf{k} = \alpha\epsilon x\mathbf{i} + \alpha\epsilon y\mathbf{j} - 2\alpha\epsilon z\mathbf{k} - B\epsilon[(\mathbf{i}\dot{x} + \mathbf{j}\dot{y} + \mathbf{k}\dot{z}) \times \mathbf{k}\ ],$$

or

$$(m\ddot{x} \cdot \alpha\epsilon x + B\epsilon\dot{y})\mathbf{i} + (m\ddot{y} - \alpha\epsilon y - B\epsilon\dot{x})\mathbf{j} + (m\ddot{z} + 2\alpha\epsilon z)\mathbf{k} = 0. \tag{8}$$

The coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ must vanish separately. Thus we have

$$m\ddot{x} - \alpha\epsilon x + B\epsilon\dot{y} = 0 \tag{9}$$

$$m\ddot{y} - \alpha\epsilon y - B\epsilon\dot{x} = 0 \tag{10}$$

$$m\ddot{z} + 2\alpha\epsilon z = 0 \tag{11}$$

3. General Solutions.

To solve for x, differentiate (9) with respect to t and solve for $\ddot{y}$,

$$\ddot{y} = -\left[\frac{m\dddot{x} - \alpha\epsilon\dot{x}}{B\epsilon}\right]$$

and substitute into (10),

$$m^2\ddddot{x} + (B^2\epsilon^2 - m\alpha\epsilon)\ddot{x} + \alpha\epsilon^2 By = 0. \tag{12}$$

Differentiate (12) with respect to x and solve for y

$$\dot{y} = -\left[\frac{m^2\dddot{\ddot{x}} + (B^2\epsilon^2 - m\alpha\epsilon)\dddot{x}}{\alpha\epsilon^2 B}\right],$$

and substitute into (9) and have

$$m^2 \ddddot{x} + (B^2\epsilon^2 - 2m\alpha\epsilon)\ddot{x} + \alpha^2\epsilon^2 x = 0 \tag{13}$$

We substitute $x = e^{\gamma t}$ into (13) and have

$$[m^2\gamma^4 + (B^2\epsilon^2 - 2m\alpha\epsilon)\gamma^2 + \alpha^2\epsilon^2]e^{\gamma t} = 0,$$

or

$$m^2\gamma^4 + (B^2\epsilon^2 - 2m\alpha\epsilon)\gamma^2 + \alpha^2\epsilon^2 = 0. \tag{14}$$

Solving for γ^2,

$$\gamma^2 = \frac{2m\alpha\epsilon - B^2\epsilon^2 \pm B\epsilon\sqrt{B^2\epsilon^2 - 4m\alpha\epsilon}}{2m^2} \tag{15}$$

Solving (15) for γ,

$$\gamma = \frac{\pm\sqrt{4m\alpha\epsilon - B^2\epsilon^2} \pm B\epsilon i}{2m} \tag{16}$$

Let

$$\gamma_1 = \frac{\sqrt{4m\alpha\epsilon - B^2\epsilon^2} + B\epsilon i}{2m} \tag{17}$$

$$\gamma_2 = \frac{\sqrt{4m\alpha\epsilon - B^2\epsilon^2} - B\epsilon i}{2m} \tag{18}$$

Since the four roots of (14) are $\gamma = \pm\gamma_1$, $\gamma = \pm\gamma_2$, four linearly independent particular solutions of (13) are $x = e^{\gamma_1 t}$, $x = e^{\gamma_2 t}$, $x = e^{-\gamma_1 t}$ and $x = e^{-\gamma_2 t}$. Thus a general solution will be

$$x = A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t} + A_3 e^{-\gamma_1 t} + A_4 e^{-\gamma_2 t}, \tag{19}$$

where A_1, A_2, A_3 and A_4 are arbitrary constants. We substitute (19) into (9) and have

$$mA_1\gamma_1^2 e^{\gamma_1 t} + mA_2\gamma_2^2 e^{\gamma_2 t} + mA_3\gamma_1^2 e^{-\gamma_1 t} + mA_4\gamma_2^2 e^{-\gamma_2 t}$$
$$- \alpha\epsilon(A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t} + A_3 e^{-\gamma_1 t} + A_4 e^{-\gamma_2 t}) + B\epsilon\dot{y} = 0,$$

or

$$(m\gamma_1^2 - \alpha\epsilon)(A_1 e^{\gamma_1 t} + A_3 e^{-\gamma_1 t}) + (m\gamma_2^2 - \alpha\epsilon)(A_2 e^{\gamma_2 t} + A_4 e^{-\gamma_2 t}) + B\epsilon\dot{y} = 0.$$

Solving for $\dot{y}$ and integrating,

$$y = \left(\frac{\alpha\epsilon - m\gamma_1^2}{B\epsilon\gamma_1}\right)\left(A_1 e^{\gamma_1 t} - A_3\, e^{-\gamma_1 t}\right) +$$
$$\left(\frac{\alpha\epsilon - m\gamma_2^2}{B\epsilon\gamma_2}\right)\left(A_2 e^{\gamma_2 t} - A_4\, e^{-\gamma_2 t}\right) + C, \tag{20}$$

where C is a constant of integration. Substituting (17) and (18) into (20),

$$y = -A_1 ie^{\gamma_1 t} + A_2 ie^{\gamma_2 t} + A_3 ie^{-\gamma_1 t} - A_4 ie^{-\gamma_2 t} + C. \tag{21}$$

By substituting (21) into (10), we see that $C = 0$ and thus we have

$$y = -A_1 ie^{\gamma_1 t} + A_2 ie^{\gamma_2 t} + A_3 ie^{-\gamma_1 t} - A_4 ie^{-\gamma_2 t}. \tag{22}$$

The solution of (11) is simple and well known. We let $\sqrt{\frac{2\alpha\epsilon}{m}} = \omega_0$ and two linearly independent particular solutions are $z = \cos \omega_0 t$ and $z = \sin \omega_0 t$. A general solution is

$$z = C_1 \sin \omega_0 t + C_2 \cos \omega_0 t, \tag{23}$$

where C_1 and C_2 are constants of integration.

Since we are only interested in real values of t, x, y and z, we find it expedient to put our solutions in a form such that x, y and z are real when t and the constants of integration are real. Equation (23) accomplishes this for z. It will however be necessary to consider x and y for special cases for different values of B.

Case I. $B = 0$.

For this case equations (9) and (10) become

$$m\ddot{x} - \alpha\epsilon x = 0, \tag{24}$$

$$m\ddot{y} - \alpha\epsilon y = 0. \tag{25}$$

The general solution is

$$x = A_1 e^{\eta t} + A_2 e^{-\eta t}, \tag{26}$$

$$y = A_1^0 e^{\eta t} + A_2^0 e^{-\eta t}, \tag{27}$$

where $\eta = \sqrt{\frac{\alpha\epsilon}{m}}$ and A_1, A_2, A_1^0 and A_2^0 are constants of integration.

Case II. $0 < |B| < \left|2\sqrt{\frac{m\alpha}{\epsilon}}\right|$.

Let $\gamma_1 = \beta + i\omega$ and $\gamma_2 = \beta - i\omega$. Thus $\beta = \sqrt{\frac{4m\alpha\epsilon - B^2\epsilon^2}{2m}}$ and $\omega = \frac{B\epsilon}{2m}$. Equation (19) becomes

$$x = A_1 e^{(\beta+i\omega)t} + A_2 e^{(\beta-i\omega)t} + A_3 e^{-(\beta+i\omega)t} + A_4 e^{(-\beta+i\omega)t}, \tag{28}$$

or

$$x = A_1 e^{\beta t} (\cos \omega t + i \sin \omega t) + A_2 e^{\beta t} (\cos \omega t - i \sin \omega t)$$

$$+ A_3 e^{-\beta t}(\cos \omega t - i \sin \omega t) + A_4 e^{-\beta t}(\cos \omega t + i \sin \omega t),$$

or

$$x = e^{\beta t}[(A_1 + A_2)\cos \omega t + (A_1 - A_2)\, i \sin \omega t]$$
$$+ e^{-\beta t}[(A_3 + A_4)\cos \omega t + (A_4 - A_3)\, i \sin \omega t] \quad (29)$$

We let

$$A_1 + A_2 = D_1, \quad (A_1 - A_2)i = D_2, \quad A_4 + A_3 = D_3, \quad (A_4 - A_3)i = D_4, \quad (30)$$

and equation (29) becomes

$$x = e^{\beta t}(D_1 \cos \omega t + D_2 \sin \omega t) + e^{-\beta t}(D_3 \cos \omega t + D_4 \sin \omega t). \quad (31)$$

In like manner equation (22) becomes

$$y = e^{\beta t}[(A_2 - A_1)i \cos \omega t + (A_1 + A_2)\sin \omega t]$$
$$+ e^{-\beta t}[(A_3 - A_4)i \cos \omega t + (A_3 + A_4)\sin \omega t] \quad (32)$$

From (30) and (32), we have

$$y = e^{\beta t}(-D_2 \cos \omega t + D_1 \sin \omega t) - e^{-\beta t}(D_4 \cos \omega t - D_3 \sin \omega t) \quad (33)$$

Case III. $B = \pm 2\sqrt{\frac{m\alpha}{\epsilon}}$.

In this case, $\gamma_1 = \frac{B\epsilon i}{2m}$, $\gamma_2 = -\frac{B\epsilon i}{2m}$. Thus $\gamma_2 = -\gamma_1$ and we have two sets of equal roots for γ. Also γ is always pure imaginary. We let $\gamma_1 = i\omega$ and the solution for x becomes

$$x = (A_1 + A_2 t)e^{i\omega t} + (A_3 + A_4 t)e^{-i\omega t}, \quad (34)$$

where A_1, A_2, A_3 and A_4 are constants of integration. We transform this into terms of another set of constants of integration.

$$x = (D_1 + D_2 t)\cos \omega t + (D_3 + D_4 t)\sin \omega t. \quad (35)$$

Likewise the solution for y may be written

$$y = (D_1^0 + D_2^0 t)\cos \omega t + (D_3^0 + D_4^0 t)\sin \omega t. \quad (36)$$

Let us solve for D_1^0, D_2^0, D_3^0 and D_4^0 in terms of D_1, D_2, D_3 and D_4. We substitute (35) and (36) into (9) and have

$$-\left(\frac{B^2\epsilon^2}{4m} + \alpha\epsilon\right)[(D_1 + D_2 t)\cos \omega t + (D_3 + D_4 t)\sin \omega t]$$

$$- D_2 B\epsilon \sin \omega t + D_4 B\epsilon \cos \omega t + D_2^0 B\epsilon \cos \omega t + D_4^0 B\epsilon \sin \omega t$$

$$-\frac{B^2\epsilon^2}{2m}\,[(D_1^0 + D_2^0 t)\sin\omega t - (D_3^{00} + D_4^0 t)\cos\omega t] = 0. \tag{37}$$

Since for case III, $B^2 = \frac{4m\epsilon}{\epsilon}$, or $\alpha\epsilon = \frac{B^2\epsilon^2}{4m}$, we so substitute into (37) and have

$$\frac{B^2\epsilon^2}{2m}\,(D_3^0 - D_1 + (D_4^0 - D_2)t)\cos\omega t + B\epsilon(D_2^0 + D_4)\cos\omega t -$$

$$\frac{B^2\epsilon^2}{2m}\,(D_1^0 + D_3 + (D_2^0 + D_4)t)\sin\omega t + B\epsilon(D_4^0 - D_2)\sin\omega t = 0 \tag{38}$$

Equation (38) must hold for all t and it will do so if and only if

$$D_1^0 = -D_3, \quad D_2^0 = -D_4, \quad D_3^0 = D_1, \quad D_4^0 = D_2. \tag{39}$$

Substituting the equations (39) into (36), we have

$$y = -(D_3 + D_4 t)\cos\omega t + (D_1 + D_2 t)\sin\omega t. \tag{40}$$

Case IV. $|B| > |\,2\sqrt{\frac{m\alpha}{\epsilon}}\,|$

For this case, we write $\gamma = \dfrac{i(\pm B\epsilon \pm \sqrt{B^2\epsilon^2 - 4m\alpha\epsilon})}{2m}$.

γ is pure imaginary and we let $\gamma = \pm\, i\omega$. We further define

$$\omega_1 = \frac{B\epsilon + \sqrt{B^2\epsilon^2 - 4m\alpha\epsilon}}{2m} \tag{41}$$

$$\omega_2 = \frac{B\epsilon - \sqrt{B^2\epsilon^2 - 4m\alpha\epsilon}}{2m} \tag{42}$$

Thus $\gamma_1 = i\omega_1$, $\gamma_2 = -\,i\omega_2$ and the solution in equations (19) and (22) becomes

$$x = A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_2 t} + A_3 e^{-i\omega_1 t} + A_4 e^{i\omega_2 t}, \tag{43}$$

$$y = -A_1 i e^{i\omega_1 t} + A_2 i e^{-i\omega_2 t} + A_3 i e^{-i\omega_1 t} - A_4 i e^{i\omega_2 t}. \tag{44}$$

Equation (43) may be written

$$x = A_1(\cos\omega_1 t + i\sin\omega_1 t) + A_2(\cos\omega_2 t - i\sin\omega_2 t) +$$

$$A_3(\cos\omega_1 t - i\sin\omega_1 t) + A_4(\cos\omega_2 t + i\sin\omega_2 t), \text{ or}$$

$$x = (A_1 - A_3)\, i\sin\omega_1 t + (A_1 + A_3)\cos\omega_1 t +$$

$$(A_4 - A_2)\, i\sin\omega_2 t + (A_2 + A_4)\cos\omega_2 t. \tag{45}$$

Let

$$(A_1 - A_3)i = D_1,\ A_1 + A_3 = D_2,\ (A_4 - A_2)i = D_3,\ A_2 + A_4 = D_4, \quad (46)$$

and (45) becomes

$$x = D_1\sin\omega_1 t + D_2\cos\omega_1 t + D_3\sin\omega_2 t + D_4\cos\omega_2 t. \quad (47)$$

Equation (44) may be written

$$y = -A_1 i(\cos\omega_1 t + i\sin\omega_1 t) + A_2 i(\cos\omega_2 t - i\sin\omega_2 t)$$

$$+ A_3 i(\cos\omega_1 t - i\sin\omega_1 t) - A_4 i(\cos\omega_2 t + i\sin\omega_2 t),\ \text{or}$$

$$y = (A_1 + A_3)\sin\omega_1 t + (A_3 - A_1)\,i\cos\omega_1 t +$$

$$(A_2 + A_4)\sin\omega_2 t + (A_2 - A_4)\,i\cos\omega_2 t. \quad (48)$$

Substituting (46) into (48),

$$y = D_2\sin\omega_1 t - D_1\cos\omega_1 t + D_4\sin\omega_2 t - D_3\cos\omega_2 t. \quad (49)$$

4. Determination of Constants of Integration.

Suppose an electron starts at point $(x_0,\ y_0,\ z_0)$ with an initial velocity $(\dot{x}_0,\ \dot{y}_0,\ \dot{z}_0)$.

We study the z direction first and find by substituting $t = 0$ into (23) that $z_0 = C_2$. Differentiating equation (23) with respect to t and substituting $t = 0$, we have $\dot{z}_0 = C_1\omega_0$, or $C_1 = (\dot{z}_0)/(\omega_0)$. Thus (23) becomes

$$z = z_0\cos\omega_0 t + \frac{\dot{z}_0}{\omega_0}\sin\omega_0 t. \quad (50)$$

This becomes

$$z = z_0\cos\omega_0 t, \quad (51)$$

for the case electron starts from rest. Equation (50) may be written

$$z = \left(\sqrt{z_0^2 + \frac{\dot{z}_0^2}{\omega_0^2}}\right)\cos(\omega_0 t - \psi), \quad (52)$$

where $\psi = \tan^{-1}\left(\dfrac{\dot{z}_0}{z_0\omega_0}\right)$.

We study the equations for x and y for the different cases for different values of B.

Case I. $B = 0$

We substitute $t = 0$ into equations (26) and (27) and find

$$x_0 = A_1 + A_2, \qquad y_0 = A_1^0 + A_2^0.$$

We differentiate equations (26) and (27) with respect to t, and substitute $t = 0$ and have

$$\dot{x}_0 = (A_1 - A_2)\eta, \qquad \dot{y}_0 = (A_1^0 - A_2^0)\eta.$$

Solving for the constants of integration,

$$A_1 = \frac{x_0}{2} + \frac{\dot{x}_0}{2\eta}, \quad A_2 = \frac{x_0}{2} - \frac{\dot{x}_0}{2\eta}, \quad A_1^0 = \frac{y_0}{2} + \frac{\dot{y}_0}{2\eta}, \quad A_2^0 = \frac{y_0}{2} - \frac{\dot{y}_0}{2\eta}.$$

Equations (26) and (27) thus become

$$x = \left(\frac{x_0}{2} + \frac{\dot{x}_0}{2\eta}\right) e^{\eta t} + \left(\frac{x_0}{2} - \frac{\dot{x}_0}{2\eta}\right) e^{-\eta t}, \tag{53}$$

$$y = \left(\frac{y_0}{2} + \frac{\dot{y}_0}{2\eta}\right) e^{\eta t} + \left(\frac{y_0}{2} - \frac{\dot{y}_0}{2\eta}\right) e^{-\eta t}, \tag{54}$$

or

$$x = x_0 \cosh \eta t + \frac{\dot{x}_0}{\eta} \sinh \eta t, \tag{55}$$

$$y = y_0 \cosh \eta t + \frac{\dot{y}_0}{\eta} \sinh \eta t. \tag{56}$$

For the case initial velocity is zero, we have

$$x = x_0 \cosh \eta t, \qquad y = y_0 \cosh \eta t, \tag{57}$$

or

$$y = \left(\frac{y_0}{x_0}\right) x. \tag{58}$$

Case II. $0 < |B| < \left|2\sqrt{\frac{\kappa a}{\epsilon}}\right|$.

Substituting $t = 0$ into equations (31) and (33), we have

$$x_0 = D_1 + D_3, \qquad y_0 = -(D_2 + D_4)$$

We differentiate equations (31) and (33) with respect to t and substitute $t = 0$,

$$\dot{x}_0 = D_1\beta + D_2\omega - D_3\beta + D_4\omega = (D_1 - D_3)\beta + (D_2 + D_4)\omega,$$

$$\dot{y}_0 = -D_2\beta + D_1\omega + D_4\beta + D_3\omega = (D_4 - D_2)\beta + (D_1 + D_3)\omega .$$

Solving for D_1, D_2, D_3 and D_4 and substituting the values obtained into equations (31) and (33)

$$x = e^{\beta t}\left[\left(\frac{\dot{x}_0 + \beta x_0 + \omega y_0}{2\beta}\right)\cos\omega t - \left(\frac{\dot{y}_0 + \beta y_0 - \omega x_0}{2\beta}\right)\sin\omega t\right]$$

$$+ e^{-\beta t}\left[\left(\frac{\beta x_0 - \dot{x}_0 - \omega y_0}{2\beta}\right)\cos\omega t + \left(\frac{\dot{y}_0 - \beta y_0 - \omega x_0}{2\beta}\right)\sin\omega t\right], \quad (59)$$

$$y = e^{\beta t}\left[\left(\frac{\dot{y}_0 + \beta y_0 - \omega x_0}{2\beta}\right)\cos\omega t + \left(\frac{\dot{x}_0 + \beta x_0 + \omega y_0}{2\beta}\right)\sin\omega t\right]$$

$$- e^{-\beta t}\left[\left(\frac{\dot{y}_0 - \beta y_0 - \omega x_0}{2\beta}\right)\cos\omega t + \left(\frac{\dot{x}_0 - \beta x_0 + \omega y_0}{2\beta}\right)\sin\omega t\right]. \quad (60)$$

Let

$$\tan\theta_1 = \frac{\beta y_0 - \omega x_0}{\omega y_0 + \beta x_0}, \quad \tan\theta_2 = \frac{\omega x_0 + \beta y_0}{\beta x_0 - \omega y_0}, \quad \tan\psi_0 = \frac{\dot{y}_0}{\dot{x}_0},$$

$r_0 = \sqrt{x_0^2 + y_0^2}$, and equations (59) and (60) become

$$x = r_0\,\frac{\sqrt{\beta^2 + \omega^2}}{2\beta} \times$$

$$[(\cos\theta_1 \cos\omega t - \sin\theta_1 \sin\omega t)e^{\beta t} + (\cos\theta_2 \cos\omega t - \sin\theta_2 \sin\omega t)e^{-\beta t}]$$

$$+ \frac{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}}{2\beta}\,(e^{\beta t} - e^{-\beta t})(\cos\psi_0 \cos\omega t - \sin\psi_0 \sin\omega t), \quad (61)$$

$$y = r_0\,\frac{\sqrt{\beta^2 + \omega^2}}{2\beta} \times$$

$$[(\sin\theta_1 \cos\omega t + \cos\theta_1 \sin\omega t)e^{\beta t} + (\sin\theta_2 \cos\omega t + \cos\theta_2 \sin\omega t)e^{-\beta t}]$$

$$+ \frac{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}}{2\beta}\,(e^{\beta t} - e^{-\beta t})(\sin\psi_0 \cos\omega t + \cos\psi_0 \sin\omega t). \quad (62)$$

Equations (61) and (62) become

$$x = r_0 \frac{\sqrt{\beta^2 + \omega^2}}{2\beta} [e^{\beta t} \cos(\omega t + \theta_1) + e^{-\beta t} \cos(\omega t + \theta_2)]$$

$$+ \frac{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}}{\beta} \sinh \beta t \cos(\omega t + \psi_0), \tag{63}$$

$$y = r_0 \frac{\sqrt{\beta^2 + \omega^2}}{2\beta} [e^{\beta t} \sin(\omega t + \theta_1) + e^{-\beta t} \sin(\omega t + \theta_2)]$$

$$+ \frac{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}}{\beta} \sinh \beta t \sin(\omega t + \psi_0), \tag{64}$$

For the case in which the initial velocity is zero, equations (63) and (64) reduce to

$$x = r_0 \frac{\sqrt{\beta^2 + \omega^2}}{2\beta} [e^{\beta t} \cos(\omega t + \theta_1) + e^{-\beta t} \cos(\omega t + \theta_2)], \tag{65}$$

$$y = r_0 \frac{\sqrt{\beta^2 + \omega^2}}{2\beta} [e^{\beta t} \sin(\omega t + \theta_1) + e^{-\beta t} \sin(\omega t + \theta_2)]. \tag{66}$$

Case III. $B = \pm 2\sqrt{\frac{m\alpha}{\epsilon}}$

Substituting $t = 0$ into equations (35) and (40), we have $x_0 = D_1$, $y_0 = -D_3$. We differentiate equations (35 and (40) with respect to t and substitute $t = 0$ and have $\dot{x}_0 = D_2 + D_3\omega$, or

$$D_2 = \dot{x}_0 + y_0\omega, \quad \dot{y}_0 = -D_4 + D_1\omega, \text{ or } D_4 = x_0\omega - \dot{y}_0.$$

Thus equations (35) and (40) now become

$$x = [x_0 + (\dot{x}_0 + y_0\omega)t] \cos\omega t - [y_0 + (\dot{y}_0 - x_0\omega)t] \sin\omega t, \tag{67}$$

$$y = [y_0 + (\dot{y}_0 - x_0\omega)t] \cos \omega t + [x_0 + (\dot{x}_0 + y_0\omega)t] \sin\omega t \tag{68}$$

Equations (67) and (68) may be transformed into

$$x = r_0 [\cos(\omega t + \theta_0) + \omega t \sin(\omega t + \theta_0)] + t \sqrt{\dot{x}_0^2 + \dot{y}_0^2} \cos(\omega t + \psi_0), \tag{69}$$

$$y \quad r_0 [\sin(\omega t + \theta_0) - \omega t \cos(\omega t + \theta_0)] + t \sqrt{\dot{x}_0^2 + \dot{y}_0^2} \sin(\omega t + \psi_0). \tag{70}$$

where

$$\theta_0 = \tan^{-1}\left(\frac{y_0}{x_0}\right), \quad \psi_0 = \tan^{-1}\left(\frac{\dot{y}_0}{\dot{x}_0}\right), \quad r_0 = \sqrt{x_0^2 + y_0^2}\ .$$

If the electron starts from rest, equations (69) and (70) become

$$x = r_0\,[\cos(\omega t + \theta_0) + \omega t\ \sin(\omega t + \theta_0)], \tag{71}$$

$$y = r_0\,[\sin(\omega t + \theta_0) - \omega t\ \cos(\omega t + \theta_0)]. \tag{72}$$

Case IV. $|B| > \left|\,2\sqrt{\frac{m\alpha}{\epsilon}}\,\right|$.

Substituting $t = 0$ into equations (47) and (49), we obtain $x_0 = D_2 + D_4$, $y_0 = -(D_1 + D_3)$. We differentiate equations (47) and (49) with respect to t and substitute $t = 0$, $\dot{x}_0 = D_1\omega_1 + D_3\omega_2$, $\dot{y}_0 = D_2\omega_1 + D_4\omega_2$. Solving for the constants of integration,

$$D_1 = \frac{\dot{x}_0 + y_0\omega_2}{\omega_1 - \omega_2}, \quad D_2 = \frac{\dot{y}_0 - x_0\omega_2}{\omega_1 - \omega_2}, \quad D_3 = -\frac{\dot{x}_0 + y_0\omega_1}{\omega_1 - \omega_2}, \quad D_4 = \frac{\omega_1 x_0 - \dot{y}_0}{\omega_1 - \omega_2}.$$

Substituting the above values of D_1, D_2, D_3 and D_4 into (47) and (49),

$$\begin{aligned} x = {} & \left(\frac{\dot{x}_0 + y_0\omega_2}{\omega_1 - \omega_2}\right)\sin\omega_1 t + \left(\frac{\dot{y}_0 - x_0\omega_2}{\omega_1 - \omega_2}\right)\cos\omega_1 t \\ & -\left(\frac{\dot{x}_0 + y_0\omega_1}{\omega_1 - \omega_2}\right)\sin\omega_2 t - \left(\frac{\dot{y}_0 - \omega_1 x_0}{\omega_1 - \omega_2}\right)\cos\omega_2 t, \end{aligned} \tag{73}$$

$$\begin{aligned} y = {} & \left(\frac{\dot{y}_0 - x_0\omega_2}{\omega_1 - \omega_2}\right)\sin\omega_1 t - \left(\frac{\dot{x}_0 + y_0\omega_2}{\omega_1 - \omega_2}\right)\cos\omega_1 t, \\ & -\left(\frac{\dot{y}_0 - \omega_1 x_0}{\omega_1 - \omega_2}\right)\sin\omega_2 t + \left(\frac{\dot{x}_0 + y_0\omega_1}{\omega_1 - \omega_2}\right)\cos\omega_2 t. \end{aligned} \tag{74}$$

Let

$$\theta_1 = \tan^{-1}\left(\frac{\dot{x}_0 + y_0\omega_2}{\omega_2 x_0 \quad \dot{y}_0}\right), \quad \theta_2 = \tan^{-1}\left(\frac{\omega_1 y_0 + \dot{x}_0}{\omega_1 x_0 \quad \dot{y}_0}\right), \quad r_0 = \sqrt{x_0^2 + y_0^2}$$

and equations (73) and (74) become

$$x = \frac{(\sin\theta_1 \sin\omega_1 t - \cos\theta_1 \cos\omega_1 t)}{\omega_1 - \omega_2}\sqrt{\omega_2^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_2(\dot{x}_0 y_0 - x_0\dot{y}_0)} +$$

$$\frac{(\cos\theta_2 \cos\omega_2 t - \sin\theta_2 \sin\omega_2 t)}{\omega_1 - \omega_2}\sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0\dot{y}_0)}, \quad (75)$$

$$y = \frac{-(\cos\theta_1 \sin\omega_1 t + \sin\theta_1 \cos\omega_1 t)}{\omega_1 - \omega_2}\sqrt{\omega_2^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\ {}_2(\dot{x}_0 y_0 - x_0\dot{y}_0)} +$$

$$\frac{(\sin\theta_2 \cos\omega_2 t + \cos\theta_2 \sin\omega_2 t)}{\omega_1 - \omega_2}\sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0\dot{y}_0)}. \quad (76)$$

Equations (75) and (76) may be written

$$x = \sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0\dot{y}_0)}\ \frac{\cos(\omega_2 t + \theta_2)}{\omega_1 - \omega_2} -$$

$$\sqrt{\omega_2^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_2(\dot{x}_0 y_0 - x_0\dot{y}_0)}\ \frac{\cos(\omega_1 t + \theta_1)}{\omega_1 - \omega_2}, \quad (77)$$

$$y = \sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0\dot{y}_0)}\ \frac{\sin(\omega_2 t + \theta_2)}{\omega_1 - \omega_2} -$$

$$\sqrt{\omega_2^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_2(\dot{x}_0 y_0 - x_0\dot{y}_0)}\ \frac{\sin(\omega_1 t + \theta_2)}{\omega_1 - \omega_2}. \quad (78)$$

If the electron starts from rest, equations (77) and (78) become

$$x = \frac{r_0}{\omega_1 - \omega_2}[\omega_1 \cos(\omega_2 t + \theta_0) - \omega_2 \cos(\omega_1 t + \theta_0)], \quad (79)$$

$$y = \frac{r_0}{\omega_1 - \omega_2}[\omega_1 \sin(\omega_2 t + \theta_0) - \omega_2 \sin(\omega_1 t + \theta_0)], \quad (80)$$

where $\theta_0 = \tan^{-1}\dfrac{y_0}{x_0}$.

5. *Interpretation of Results.*

5.1 *The z Component of the Motion.*

Either equations (50) or (52) show that the z-component of the tion is simple harmonic motion. The motion will be total simple harmonic motion only in case the electron starts on the z-axis with initial velocity only in the z direction or zero initial velocity. Otherwise there will be a component in the xy-plane to add to the motion in the z direction.

However if the electron is placed in the plane $z = 0$ with no initial velocity in the z direction, the electron will stay in this plane. If placed at the origin with zero initial velocity the electron will stay at the origin.

5.2 *The Component of the Motion in the xy-plane.*

5.21 Case I. $B = 0$

(a) Initial Velocity Zero.

As shown by equation (58), the component of the motion in the xy-plane is a straight line. Thus the motion will always be in a plane thru the z-axis and defined by $y = \dfrac{y_0}{x_0}\, x$. That is the trajectory is a plane curve. It will resemble a sine curve. The amplitude will remain constant but the distance between each peak will continually increase.

(b) Initial Velocity $\neq 0$.

The component of the motion in the xy-plane is given by equations (53) and (54) or by (55) and (56) and is a straight line only when $\begin{vmatrix} x_0 & \dot{x}_0 \\ y_0 & \dot{y}_0 \end{vmatrix} = 0$. Otherwise the component of the trajectory in the xy-plane will be a curve that tends asymptotically to $y = \left(\dfrac{\eta y_0 + \dot{y}_0}{\eta x_0 + \dot{x}_0} \right) x$. The electron continually proceeds towards infinity with continually increasing velocity.

5.22 Case II. $0 < |B| < \left|2\sqrt{\frac{m\alpha}{e}}\right|$.

(a) Electron Starts from Rest.

The component of the trajectory in the xy-plane is defined by equations (65) and (66). It is seen that the position of the electron in the xy-plane may be represented by the sum of two vectors that rotate with the same constant angular velocity. The length of one increases exponentially with time and the length of the other decreases exponentially with time.

(b) Initial Velocity $\neq 0$.

The component of the motion in the xy-plane is given by equations (63) and (64). The position of the electron in the xy-plane is represented by the vector that represents its position when starting from rest plus another vector that rotates with the same constant angular velocity and whose magnitude is $\sqrt{\dfrac{\dot{x}_0^2 + \dot{y}_0^2}{\beta}}\ \sinh \beta t$.

5.23 Case III. $B = \pm 2\sqrt{\dfrac{m\alpha}{\epsilon}}$

(a) Initial Velocity Zero.

The position of the electron in the xy-plane is given by equations (71) and (72). This component of the trajectory is the involute of a circle and can be traced by a fixed point on a straight line that rolls without slipping on a circle, the circle in question having its center at the origin and the electron starts at a point on the circumference. The projection of the path on the xy-plane is of course a spiral.

(b) Initial Velocity $\neq 0$.

The projection of the position of the electron on the xy-plane is given by equations (69) and (70) and is represented by the sum of the vector that represents its position when starting from rest and a vector that rotates with constant angular velocity and radiates from the origin with constant velocity. The component of the motion in the xy-plane for this case will also be a spiral except when $\psi_0 = \theta_0 + \dfrac{\pi}{2}$ and $\omega = \sqrt{\dot{x}_0^2 + \dot{y}_0^2}$. In this case the motion in the xy-plane will be a circle.

5.24 Case IV. $|B| > \left|2\sqrt{\dfrac{m\alpha}{\epsilon}}\right|$

This is the only case where the electron does not continually diverge towards infinity.

(a) Initial Velocity Zero.

The component of the motion in the xy-plane will be given by equations (79) and (80). The projection of the position of the electron on the xy-plane is represented by the sum of two vectors of constant magnitude that rotate with constant angular velocity. This component of the trajectory is an epicycloid and can be traced by a fixed point on a circle that rolls without slipping on a fixed circle. The fixed circle has center at origin and starting point of electron on circumference. The radius of the rolling circle is given by $(r_0\,\omega_2)/(\omega_1 - \omega_2)$. The component of the motion in the xy-plane will

be periodic if $\frac{\omega_1}{\omega_2}$ is rational. It will be almost periodic in either case $\frac{\omega_1}{\omega_2}$ is irrational or rational.

The projection of the position of the electron on the xy-plane will always be inside or on a ring bounded by two concentric circles of radii r_0 and $\frac{r_0(\omega_1 + \omega_2)}{\omega_1 - \omega_2}$. For the total motion the electron will always be inside or on the boundary of a region bounded by concentric circular cylinders of radii, r_0 and $\frac{r_0(\omega_1 + \omega_1)}{\omega_1 - \omega_2}$ and the two planes, $z = \pm z_0$.

(b) Initial Velocity $\neq 0$

The projection of the position of the electron on the xy-plane is given by equations (77) and (78) and can again be represented as the sum of two vectors of constant magnitude that rotate with constant angular velocity. The component of the trajectory in the xy-plane need not necessarily be an epicycloid but is an epitrochoid and is traced by a carried point not necessarily on the circumference of a circle that rolls without slipping on the exterior of a fixed circle. To find the radii of the rolling and fixed circles, we note that

$$r_f \omega_2 t = r_r(\omega_1 - \omega_2)t, \text{ or}$$

$$r_f \omega_2 = r_r(\omega_1 - \omega_2), \tag{81}$$

where r_f denotes the radius of the fixed circle and r_r the radius of the rolling circle. Also

$$r_r + r_f = \frac{\sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0 \dot{y}_0)}}{\omega_1 - \omega_2} \tag{82}$$

From (81) and (82), we find that

$$r_f = \frac{\sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0 \dot{y}_0)}}{\omega_1}. \tag{83}$$

$$r_r = \frac{\omega_2 \sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0 \dot{y}_0)}}{\omega_1(\omega_1 - \omega_2)}. \tag{84}$$

The component of the motion in the xy- plane will again be periodic if and only if $\frac{\omega_1}{\omega_2}$ is rational and it will be almost periodic whether $\frac{\omega_1}{\omega_2}$ is rational or irrational.

The projection of the electron on the xy-plane will always be inside or on a ring bounded by two concentric circles of radii

$$\frac{\sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0\dot{y}_0)} + \sqrt{\omega_2^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_2(\dot{x}_0 y_0 - x_0\dot{y}_0)}}{\omega_1 - \omega_2} \quad \text{and}$$

$$\frac{\sqrt{\omega_1^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_1(\dot{x}_0 y_0 - x_0\dot{y}_0)} - \sqrt{\omega_2^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_2(\dot{x}_0 y_0 - x_0\dot{y}_0)}}{\omega_1 - \omega_2}.$$

as this represents the sum and difference of the magnitudes of the two rotating vectors. The electron will always be inside or on the boundary of a region bounded by two concentric right circular cylinders of the above radii with the z-axis as center and bounded above and below by the two planes,

$$z = \pm\sqrt{z_0^2 + \frac{\dot{z}_0^2}{\omega_0^2}}$$

5.241 The Motion as $B \to \infty$.

We divide the numerator and denominator of the right side of (77) by ω_1 and have

$$x = \frac{\cos(\omega_2 t + \theta_2)}{1 - \frac{\omega_2}{\omega_1}}\sqrt{r_0^2 + \frac{\dot{x}_0^2 + \dot{y}_0^2}{\omega_1^2} + \frac{2}{\omega_1}(\dot{x}_0 y_0 - x_0\dot{y}_0)} -$$

$$\frac{\frac{\cos(\omega_1 t + \theta_1)}{\omega_1}}{1 - \frac{\omega_2}{\omega_1}}\sqrt{\omega_2^2 r_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + 2\omega_2(\dot{x}_0 y_0 - x_0\dot{y}_0)}. \tag{85}$$

From (41) and (42), we have

$$\omega_1\omega_2 = \frac{\alpha\epsilon}{m}. \tag{86}$$

$$\omega_2 = \frac{\alpha\epsilon}{\omega_1 m}. \tag{87}$$

From (41), it is seen that as $B \to \infty$, $\omega_1 \to \infty$, and from (87), it follows that $\omega_2 \to 0$. Thus (85) tends to

$$x = r_0 \cos \theta_2 \tag{88}$$

Now

$$\tan \theta_2 = \frac{\omega_1 y_0 + \dot{x}_0}{\omega_1 x_0 - \dot{y}_0} = \frac{y_0 + \dfrac{\dot{x}_0}{\omega_1}}{x_0 - \dfrac{\dot{y}_0}{\omega_1}}$$

Thus as $B \to \infty$ and $\omega_1 \to \infty$, $\tan \theta_2 \to \dfrac{y_0}{x_0}$ and $\cos \theta_2 \to \dfrac{x_0}{r_0}$, and consequently (88) tends to

$$x = x_0 \tag{89}$$

In like manner as $B \to \infty$, (78) tends to

$$y = y_0 \tag{90}$$

The result in equations (89) and (90) may be stated more precisely: For every real $\eta > 0$, and for every real and positive T, there exist an $M(\eta, T)$ such that when $|B| > M$ and $t < T$, then the position of the electron in the xy-plane is always inside a circle of radius η about (x_0, y_0). The total motion will always be inside a cylinder of radius η and center the line $x = x_0$, $y = y_0$, bounded above and below by the planes,

$$z = \pm \sqrt{z_0^2 + \frac{\dot{z}_0^2}{\omega_0^2}}$$

when $|B| > M$ and $t < T$.

6. Acknowledgements.

The author wishes to thank Dr. L. E. Loveridge for having suggested the problem and the various subtopics discussed. The research was done under his supervision. The author also wishes to thank Mr. William G. Poelstra for having proofread earlier editions of the paper.

Woodburn, Kentucky.

CURRENT PAPERS AND BOOKS

Edited by H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

* * * * * * * * * *

This comment deals with the apparent paradox in the paper, *On a Certain Problem in Mechanics* by Murray R. Spiegel, in the November-December 1956 issue.

There is no paradox. If one proceeds from his equation

$$(5) \qquad v = 4 - 2x$$

from which one concludes that v is zero when x is 2, to integrate the equivalent equation,

$$(5.1) \qquad \frac{dx}{dt} = 4 - 2x$$

we get, after applying the initial conditions,

$$(5.2) \qquad x = 2(1 - e^{-2t}).$$

From this we see that as $t \to \infty$, $x \to 2$.
This may be derived from his equation

$$(6) \qquad v = 4e^{-2t}$$

since, integrating the equivalent equation

$$(6.1) \qquad \frac{dx}{dt} = 4e^{-2t}$$

and using the initial conditions also gives (5.2).

That is, as $t \to \infty$, we have $x \to 2$ and $v \to 0$, so it is not incorrect to state that $v = 0$ when $x = 2$.

J. Lawrence Botsford,
University of Idaho.

(Continuation of *Some Remarks Concerning a Comment*, by Homer V. Craig, Vol. 30, No. 4, March-April 1957)

Actually, the startling situation which L S L brings to attention, aside from its own intrinsic interest, might very well serve simultaneously to point up the need for careful thinking in work involving infinite aggregates and processes and to provide an introduction to some fascinating mathematical topics. As an instance let us consider some of the properties of the set of points common to AC and the set of stair lines. We shall call these points contact points, thus P is a contact point if P is on some stair line and also on the line AC. Now the set of contact points has the interesting and important property of being "dense". This means that if P and Q are two distinct contact points, then there are infinitely many contact points between P and Q. To verify this it suffices to note that the process of constructing the n + 1st stair line calls for the selection of three new contact points between each pair of adjacent contact points ${}_nP_i$, ${}_nP_{i+1}$ of the nth stair line and the process is continued without end. Consequently, if we attempt to mark on a piece of paper physical counterparts of our abstract (or mathematical) points, we would find that the result would be a physical representation of the abstract line AC. If on the other hand we turn back to the abstract set of contact points, we see that the situation is very different. To examine this matter further we first recall that a well known postulate concerning points on a line and real numbers allows us to assert in the present case that if d is a real number and $0 \overline{<} d \overline{<} 2$, then that is a unique point P on AC such that the distance AP is d. But the distance from A to the ith contact point of the nth stair line turns out to be $(i - 1)/2^{2n-3}$ and these are very special numbers of the type a/b with a and b integers the so called rational numbers. Furthermore, it develops that there are infinitely many numbers between 0 and 2 which cannot be expressed in the form a/b with a and b integers. Thus in the abstract or mathematical case the set of contact points is dense but does not fill the line. The set of non-contact points is also dense. Hence the question arises: Is it possible to devise a means for "measuring" or "weighing" the infinite set of contact points and the infinite set of non-contact points? Is there any sense in which one of these infinite sets could be described as richer than the other?

One possible method of distinction between these two sets involves the concept "countably infinite" or "denumerable". A set is denumerable if it is possible by some scheme or another to set up what might be called a monogamous mating of the individuals of the set with the infinite set of positive integers or counters: 1,2,3,....

Specifically, the question in the present case is: Can a procedure be specified which will mate a certain single contact point to the counter 1, another single contact point to the number 2, etc., and have the property that if P is a contact point, then P will be correlated to some counter. A moments consideration will show that this can be done. Evidently, we can mate A to 1, C to 2, ${}_2P_2$ to 3, ${}_2P_3$ to 4, ${}_2P_4$ to 5 and then pass to the third stair line and begin with ${}_3P_2$ and proceed toward C omitting all of the contact points which have previously been counted. Next the process would be continued with the following stair line (number four), etc., without end. Any particular contact point that might be selected is point ${}_nP_i$ for a certain choice of the integers n and i and since for $m \leqq n$ the set of contact points on stairline m is finite this point ${}_nP_i$ would eventually be reached by this process.

It is interesting to observe, however, that in contradistinction to our experience with finite aggregates, we can not proceed by correlating A to 1, and the contact point next to A to 2 etc.; because there is no contact point Q which may be described as next to A since there are infinitely many contact points between A and Q for Q not A.

We now turn to the set of non-contact points and make use of a theorem which asserts that the set of irrational numbers between 0 and 1 is *uncountably infinite*. Hence it follows that the set of non-contact points must have the same property, for the set of distances of non-contact points from A includes all of the irrational numbers between 0 and 2.

A second well known procedure which is involved in some of the various methods for measuring sets of points when applied to the set of contact points leads to an astonishing situation. The procedure in the abridged form in which we shall apply it may be described as follows: (1) Having mated the contact points to the counters, we select a positive number ϵ (say ϵ = 1/million) and cover the contact point mated to 1 with an interval of length $\epsilon/2$, then we cover the contact point mated to 2 with an interval half as large as the previous one, i.e., of length $\epsilon/4$, and regard the process as being continued without end. If we add the lengths of these intervals as we go along we shall be led to the infinite series

$$\epsilon/2 + \epsilon/4 + \epsilon/8 + \ldots \epsilon/2^n + \ldots .$$

Here we note that if we stop at any given term while adding, then that term itself is just the number that needs to be added to the sum to bring the total to ϵ, but the next erm after the given term is one-half of the given term. To illustrate, if we stop with the second term, the sum in $\epsilon/2 + \epsilon/4$ or $\epsilon - \epsilon/4$ while the third term is $\epsilon/8$. Hence if S_n denotes the sum of the first n terms then re-

gardless of the size of the integer n, $S_n < \epsilon$. But on the other hand any contact point that may be designated will be covered with an interval by this process.

It now becomes apparent that if we start to study various questions arising in connection with the sequence of stair lines that we shall be led to investigate such matters as infinite series and sequences which in turn will lead us back to a study of the real number system itself. If in addition, on AB as a base we construct a rectangle by: (1) extending CB to a point D (on the "B-side of C"), (2) drawing an equal and similarly directed line AE, and (3) drawing the line ED, then we may compare the area "under AC", i.e., the area os the figure $ACDE$ with the areas under the various stair lines and their limit. This will be an initial step leading to the concept Riemann integral. To sum up the sequence of stair lines introduced in L S L could be used as an approach to quite a variety of fascinating and useful mathematics.

Introduction to the Geometry of Complex Numbers. By Roland Deaux. Translated from the revised French edition by Howard Eves. Frederick Ungar Publishing Company, 1956, 208 pages. $6.50.

The chief purpose of this book is to explain the use of complex numbers as a tool for geometric investigation. Although the author has kept the book in the realm of pure geometry, the text has frequently been used in Europe for a course for electrical engineers. The first chapter develops the basic ground-work, discusses the algebraic forms of the fundamental planar transformations, and closes with a treatment of cross-ratio. The second chapter applies the techniques to the first chapter to plane analytic geometry, with special consideration of cycloidal and unicursal curves. The third chapter is devoted to a detailed treatment of the direct circular transformations, or homographies, and the indirect circular transformations, or antigraphies. The book contains unusually fine sets of exercises inserted after the various sections of the three chapers. This translation of a highly successful European test should be a welcome addition to the English literature on the subject. Not only is the book valuable to the student of geometry, but it is an excellent collateral text for courses in the theory of functions of a complex variable. Only elementary prerequisites are needed for reading the self-contained exposition.

Ruth Seldon

Introduction to Finite Mathematics. By John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson. Prentice Hall Inc. 70 5th. Ave., New York City, New York.

" A few years ago, the department of mathematics at Dartmouth College decided to introduce a different kind of freshman course which students could elect along with (the) more traditional ones. The new course was to be designed to introduce a student to some concepts in modern mathematics early in his college career. While primarily a mathematics course, it was to discuss applications to the biological and social sciences, and thus provide a point of view, other than that given by physics, concerning the uses of mathematics." *From the authors' preface.*

Written expressly to service the new approach to introductory college mathematics briefly outlined above, *Introduction to Finite Mathematics* gives better balance to the student's orientation to both mathematics and its applications.

An Introduction to Operations Research. Edited by C. West Churchman, Russell L. Ackoff, and E. Leonard Arnoff. John Wiley & Sons, Inc. 440 Fourth Ave. New York 16, N.Y. 645 pages, $12.00 Jan. 1957.

The essential background for evaluating the field and for understanding its potentialities and procedures is supplied in this book.

As a guide for prospective consumers and practitioners of operations research, the new book makes a basic survey that stimultates the acquisition of further knowledge and competence. The volume is illustrated with case examples from business and industry, and emphasizes the importance of defining management problems in terms of objectives, as well as the importance of administration.

Beginning with a comprehensive introduction that defines the meaning and characteristics of O. R., the book goes on to discuss "The Problem," with chapters on analysis of the organization, formulation of the problem, and weighting objectives. Successive sections cover the construction and solution of the model, inventory models (elementary and those with price breaks and restrictions), allocation and waiting-time models, replacement and competetive models, and testing, control, and implementation. Selection, training, and organization form the bulk of the closing section.

TEACHING OF MATHEMATICS

Edited by Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

Two and two are not always four.

R. Lariviere

Not all addition is as simple as two and two make four. Many a student has found this out when learning the addition of vectors by the parallelogram law. To understand this type of addition one must recall that the algebraic sum of the projections of the segments of a broken line $P_1 P_2 P_n \ldots$ upon any other line L, figure (1), is equal to the projection upon L of the line joining the initial point P_1 to the terminal point P_n

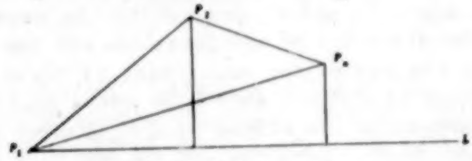

Figure (1).

A vector quantity may be defined as one which can be represented in magnitude and direction by directed line segments of the same direction and proportional magnitude. Any such segment may be called the vector of the vector quantity. Furthermore if the chosen line segment is referred to any coordinate axes, rectangular or oblique, the vector quantity is such that it can produce effects in the direction of the axes measured by the projections of its vector upon the axes.

Figure (2).

Two vectors in the same or opposite directions may be drawn in the same line and their composition obeys the laws of addition of signed numbers. If the vectors are not in the same direction

there is a well extablished experimental law, the parallelogram law for the addition of vectors, which states that the diagonial vector P_1P_3, figure (3), is the sum of vectors v_1 and v_2.

Since any two sides of a triangle are greater than the third side, P_1P_3 is clearly not an arithmetic sum of the magnitudes of v_1 and v_2. The vectors v_1, v_2 may indeed be added, but only in the sense that their projections on the same line may be added. (These projections, being in the same or opposite directions, may be added arithmetically.) It is in this sense, and not in the sense of addition of magnitudes, that we can say that P_1P_3 is the sum of the vectors v_1, v_2. We must note that, since v_1 and v_2 are not in the same direction, their effect in the direction of any line L is not the sum of their magnitudes but the sum of the magnitudes of their projections on L. The projection of P_1P_3 on line L is equal to that of v_1 plus that of v_2. Consequently the effect of P_1P_3 in every direction is equivalent to that of v_1 plus that of v_2, and P_1P_3 may reasonably be called the sum of v_1 and v_2.

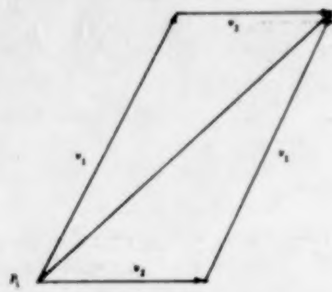

Figure (3).

The converse of the parallelogram law is also experimentally established. That is, we may replace a given vector P_1P_n ay any two components meeting at P_1 and terminating at P_1 and P_n, Figure (4). The components are equivalent to the oblique projections of P_1P_n upon oblique axes parallel to P_2P_1, P_2P_n. (We need not content ourselves with two components for P_1P_n since the projection of P_1P_n on any line is equal to the sum of the projections of the broken line P_1P_2, P_2P_3, $P_3P_4 \ldots P_{n-1}$, P_n.

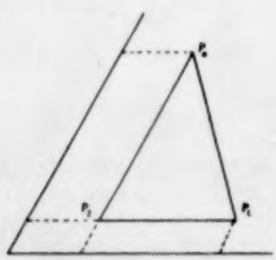

Figure (4)

To observe a vector quantity in action, let us consider two attributes of a trunk, its volume and its weight. The volume of a trunk is not a vector quantity but its weight is a vector quantity. If the trunk lies on a ramp of sufficiently steep slope the vector peculiarities of its weight become apparent. The vertical vector of its weight is replaced in action by two projections (components), one L_1 upon the surface line of the ramp, the other L_2 upon the line at right angles to this surface, figure (5). The first component then produces the motion of the trunk, the second is arrested by the reaction opposite to it of the material of the ramp.

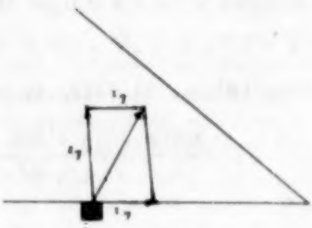

Figure (5)

The trunk moves as though the force of gravity were pulling along the ramp instead of vertically. The vector quantity, the weight, is able to accommodate itself to the situation and to lower the trunk vertically by doing so indirectly. Nothing comparable to this can happen to the other attribute, the non vector volume of the solid trunk.

A world without vector quantities that can produce effects in any direction except the reverse of their own would make a good subject for science fiction, but is hard to imagine.

Air travel has made the composition of vectors somewhat familiar to the general public. Nearly everyone now knows what is meant by the heading, the air speed, the ground speed, the course, etc. To obtain the ground spped, the speed of the wind is not added to the speed of the plane (unless it is directly a tail wind), but the projection upon any axis of the vector representing the velocity of the wind added to the projection on the same axis of the vector representing the velocity of the plane gives the projection on that axis of the vector of the resultant velocity with respect to the ground. The essential idea is that "old fashioned" addition of magnitudes can still take place if it is performed on the projections, since these are along the same line - where two and two still make four.

University of Illinois in Chicago.

Formulas for Square Roots

Stephan Nytch

Let $\sqrt{A} = C + (\sqrt{A} - C)$ $\quad A$ = a whole positive number.

$0 < \sqrt{A} - C \leq 1$ $\quad C$ = an estimate of $\sqrt{A}$, a whole number.

Examples:

$A = 3, \quad C = 1, \quad (\sqrt{A} - C) = 0.73 \ldots$

$A = 4, \quad C = 1, \quad (\sqrt{A}G\ C) = 1.$

Knowling C we find a formula for $\sqrt{A} - C$ in this way:

$$A = C^2 + 2C(\sqrt{A} - C) + (\sqrt{A} - C)^2 \tag{1}$$

By subtracting C^2 from (8) and dividing it by $A - C^2$ we get

$$1 = \frac{2C(\sqrt{A} - C) + (\sqrt{A} - C)}{A - C^2} \tag{2}$$

$A - C^2 \neq 0$ (if $A - C^2 = 0$, then we would get '$1 = \frac{0}{0}$).

Our formula for $\sqrt{A} - C$ is

$$\frac{\sqrt{A} - C}{1} = \frac{A - C^2}{2C + (\sqrt{A} - C)}$$

and

$$\sqrt{A} = C + \frac{A - C^2}{2C + (\sqrt{A} - C)} \tag{3}$$

We know, that $\sqrt{A} - C$ is always irrational except when it is 1 and it lies between 0 and 1, therefore we choose a middle value, 1/2, and put it instead of $\sqrt{A} - C$ in (3) thus

$$A = C + \frac{A - C^2}{2C + 1/2} \tag{4}$$

This formula gives a good approximation for $\sqrt{A}$. The true value of the fraction lies between $\frac{A - C}{2C}$ and $\frac{A - C^2}{2C + 1}$

Therefore the denominator $2C + (\sqrt{A} - C)$ is always smaller than $2C + 1$ and the error in $2C + 1/2$ is smaller than 1/2.

Example: To find $\sqrt{18}$; we take $C = 4$, the denominator $2 \times 4 = 8$. Add to it ½; $18 - 4^2 = 2$, so $\sqrt{18} = 4(2/8½)$ (error $= 0{,}00073 \ldots$).

New York, N.Y.

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BETWEEN THE DEVIL AND THE DEEP BLUE SEA*

Joseph Seidlin

We are all aware of the recent wave of excitement created in part by a little book with the very fetching title of "Why Johnny Can't Read." Though some sense and some nonsense intermingled freely; though attacks and counter-attacks followed each other in rapid and bewildering succession, and tensions in this era of anxious living mounted higher and higher; the schools seem to have weathered the storm. And it may be that as a result of this tumult, relatively fewer Johnnies will be among those who can't read. It maybe, that is, that those who teach reading will have improved because in self defense they were obliged to reassess their work.

Any day now, someone with a gift for turning a phrase and a conscience unruffled by scruples, will produce a bestseller with some catchy title such as, "Johnnies Multiply (in the biblical sense) All Right, But They Sure Can't Add." When that happens, it will matter but little to call attention to the fact that the teaching of mathematics has been attacked by experts these many years and decades and centuries.

But for the moment I should like to center our attention not so much on the present plight of the teaching of mathematics as on some of the conflicting forces and/or principles which for almost half a century have contributed to what some people regard, and call, a *national weakness*. What I am about to say is in the nature of an account rather than a tirade.

Over and over again we are reminded that perhaps not in the recorded history of man has there been such a stampede, - a thunderous clatter of little feet over-running the halls of learning (which is euphonism for enrollment bulges in schools). A friendly or realistic critic of our secondary schools might, while pointing out all the imperfections, marvel at the schools' stamina for survival.

Perhaps under a less humane government a surplus of children might have been disposed of, either by plowing them under - we did that to little pigs - or by some 90% parity arrangement, putting the ever increasing number of children in a kind of cold storage until such time as more and better prepared teachers and more and better constructed classrooms became available. But even if an improbable reactionary congress had passed such laws, a likely liberal Supreme Court would have declared them unconstitutional. So some sort of solution had to come from a radically different source.

* Read at the Sixth Annual Meeting of the Association of Mathematics Teachers of New York State, Syracuse, New York.

Enter the psychologists flanked by a new brand of philosophers; - another illustration of necessity being the mother of invention. since even at the secondary school level it was becoming increasingly more difficult to eliminate children afflicted by academic unfitness, more and better means of keeping these children academically alive had to be invented. In medicine the search was on for cures or prevention of diseases hitherto considered as fatal or incurable. In education the search was on for ways of teaching those who were hitherto considered unteachable.

Theories and laws (?) of learning came into existence at a feverishly accelerated rate. A rugged American philosophy of education was spreading like wildfire. During the period of the growing pains of the psychologies of education, all sorts of careless assumptions, unwarranted conclusions, and exaggerated claims were creating in the minds of the conservative a beligerent hostility toward educational psychology and educational philosophy. Much of the "Devil" emanated from this brew concocted by research (?) in the theories of learning. Gradually, however, there evolved a middle, a sort of liberal-conservative, group of school people, liberal in the sense that they were willing to examine the more mature contributions of psychologists; conservative in the sense that they were unwilling to part with some long established practices and principles. Teachers of mathematics generally belong to this buffer group. Hence they are maligned by the Progressives; they are accused of treason by the Ultra-conservatives. They are between the Devil and the Deep Blue Sea.

At the elementary and secondary school levels the teacher's job is to employ his vehicle of instruction, be it arithmetic, algebra, geometry, trigonometry, or some combination of these, so as to make a significant contribution to the overall development of a maximal number of his pupils - be they fast learners of slow learners. The time is not far off when that will be the job of college professors of mathematics as well.

Recently publicized "studies" to the contrary, most teachers of mathematics have chosen mathematics as their vehicle of instruction because they feel, and perhaps are convinced, that in our kind of world the understanding and use of mathematical concepts and processes are essential both to the individual and the society of which he is a part. "Reliable sources" inform us that Russia is outstripping us in the production of engineers. It seems that quantitatively, at least, we are falling behind. How catastrophic such a trend may become is a matter of conjecture. It may be that a few great scientists outweight many run-of-the-mill engineers. Perhaps we need to be cautioned to exert ever greater effort to provide the proper climate for the growth and development of great scientists. To fall behind

Russia in the "production" of great scientists as well as engineers may really become a very serious matter. Certainly, no sane person questions the values of mathematics in the multifaceted areas of our national strength.

But it is a sign of childish arrogance to claim a monopoly of for mathematics or any branch of human learning. While some of my best friends are mathematicians or teachers of mathematics, life would be dull indeed if there were none but mathematicians and teachers of mathematics. Professor Keyser used to say that he found more charm and reasonableness in mathematics than in mathematicians.

But as we know, many people and pupils find no charm in either mathematics or teachers of mathematics. We may comfort ourselves by saying that there is not a subject in the school curriculum free from dislike. And when critics say that mathematical incompetence is a national disgrace, we may counter by asking, why pick on mathematics? How about reading, writing, geography, history, English? Plenty of incompetence to suit any critical taste. And so there is, but that is not our problem. What can teachers of mathematics do, to achieve relatively high competencies in more and more of their pupils? It seems that making mathematics *meaningful* doesn't solve the problem. I am convinced that the present generation of teachers of mathematics are more conscious of the need for making mathematics meaningful to their pupils than any preceding generation. Let us not forget - though all too often we seem to - that understanding is only part of learning and that there is a veritable hierarchy of understandings, or, as I like to say it, that rigor is in part a function of the rigoree. Also, quite obviously, finding fault with what teachers at lower levels do, does not produce mathematical competencies in pupils. It may be that despite all the efforts of psychologists to find shortcuts for effort in the learning process, *learning is still essentially had labor.* In any sequential subject like mathematics the effort expended on learning must be not only commensurate with the inherent difficulties of the subject but must also be continuous and consistent. It is a perversion of Thorndike's Law of Effect and Dewey's dictum of Learning to do by Doing, to combine the two into a popular but psychologically unsound practice of "learning to do *by doing as little as possible*". I am convinced that when "investigators" uncover new and greater varieties of ignorance, it is not so much one or another kind of *method* of teaching that is to blame. It is rather a much mutilated pseudo philosophy of education that seems to have affected even the grass roots of our society, to wit: not only are there royal roads to learning, but every child is a prince.

However, it is still a teacher's main responsibility to facilitate learning. In the case of the brighter students the emphasis may be on accelerating learning; in the case of the duller students it may

mean initiating or encouraging learning. Psychologists are agreed that learning is self-activity (though they seem to stress "self" almost to the exclusion of activity"). To learn somebody is therefore not only grammatically incorrect, but it is psychologically unsound. to teach somebody is not only grammatically correct, but it is the only responsibility of a teacher which cannot be waived.

As a teacher, as a teacher of teachers, I subscribe unequivocally to the following propositions:

1. One who does not know his subject cannot teach it.
2. Knowing one's subject is a necessary but not a sufficient condition for teaching.
3. At all levels a teacher is morally obligated to keep on learning at least as much of his subject as is *relevant* to his teaching.
4. At all levels a teacher is morally obligated to learn and keep on learning the nature of the essential processes involved in the teaching-learning situation.
5. Until such time as moral obligation becomes all compelling, we must all work for and then relay upon mandated requirements both for initial certification and for continued validation of the appropriate certification.

Teaching of mathematics, like life itself, is an on-going process. The more we learn, the more there is to learn; the more effective we become as teachers, the more it seems conditions beyond our control make us appear less proficient. But it's fun. Let's fight it out. Let's fight it out by resisting the seductive enticements of the left-wingers and by outliving the good-old-timers.

I owe much to many teachers. My gratitude to two of them especially is almost reverential. Both of them were masters of poetic prose. I am referring to J. Cassius Keyser and David Eugene Smith. I never dare paraphrase them. Occasionally I quote one or the other. It seems that David Eugene Smith anticipated the present struggle between those who preach that mathematics can be learned some easy way, and those who claim that God made mathematics perfect, but hard, and that's that:

" Mathematics is a mountain. Vigor is needed for its ascent. The views all along the paths are magnificent. The effort of climbing is stimulating. A guide who points out the beauties, the grandeur, and the special places of interest commands the admiration of his group of pilgrims. One who fails to do this, who does not know the paths, who puts unnecessary burdens upon the pilgrim, or who blindfolds him in his progress, is unworthy of his position. The pretended guide who says that the painted panorama, seen from the rubber-tired car, is as good as the view from the summit, might better be guided than be a guide. The mountain will stand; it will not be used as a mere commerical

quarry for building stone; it will not be affected by pellets thrown from the little hillocks about; but its paths will be freed from unnecessary flints, they will be straightened where this can advantageously be done, and new paths on entirely novel plans will be made as time goes on, but these paths will be hewed out of rock, not of the froth and foam of clouds. Every worthy guide will assist in all efforts at betterment, and will urge the pilgrim at least to ascend a little way because of the fact that the same view cannot be obtained from gutters; but he will not take seriously the efforts of the quack doctor of educational pills; nor will he listen with more than passing interest to him who proclaims the sand heap to be a Matterhorn".

Alfred University, Alfred, N.Y.

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February 15, 1957

Dear Mr. James:

I enclose check for my sponsor's subscription delayed by absence from the city.

I am much impressed by the editorial balance of your current issue, Vol. 30, No. 3 which satisfies my conception of what a mathematics magazine should be. It is so unfortunate that professional magazines in every field tend to become, to speak frankly, not media for the advancement of knowledge so much as media for the advancement of careers. The professional obligation to publish should be divorced from the larger social obligation to communicate for the true advancement of knowledge. The literal meaning of "philosophy" is the love of knowledge, and a mathematics magazine should therefore inspire and feed the love of mathematical knowledge. This requires an editorial balance between material which enriches the reader's perception of the subject as a whole (covered by your two lead stories on Mathematics and Reality); material which informs; material which relates mathematics to the current social scene (The Computer's Challenge to Education); and material which stimulates and entertains.

Sincerely yours,

Alexander W. Ebin

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PROBLEMS AND QUESTIONS

Edited by Robert E. Horton, Los Angeles City College.

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

306. *Proposed by V.F. Ivanoff, San Francisco, California.*

A student drew a line segment, AC, four inches long. He constructed the mid point B so that $AB = BC = 2$ inches. He reasoned that if he removed the segment BC he would have left a segment which has no end point and is clearly not congruent with BC. This seemd to contradict his hypothesis that B was the mid point of AC. Wherein lies the student's dilemma? What would be the case if AC were replaced by a circle?

307. *Proposed by E.P. Starke, Rutgers University.*

If a number n is of the form $n = \sqrt{a+1} + \sqrt{a}$ with a rational, prove that every integral power of n is of the same form.

308. *Proposed by James Mc Cawley Jr., Chicago, Illinois.*

Find a swries expansion for a function $f(x)$ which has the property that $f(f(x))$ $g(x)$ where $g(x)$ is a given function.

309. *Proposed by Victor Thebault, Tennie, Sarthe, France.*

Determine the perfect cubes terminated on the right by nine digits 1 (or by nine digits 3).

310. *Proposed by Chih-yi Wang, University of Minnesota.*

Show that

$$\int_0^{\pi/2} \sqrt{\cos x}\, dx = \int_0^{\pi/2} \sqrt{\sin x}\, dx = 2\sqrt{2}\, E(1/\sqrt{2}) - \sqrt{2}\, F(1/\sqrt{2})$$

where $E(k)$ and $F(k)$ are elliptic integrals defined respectively by,

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2 t}\; dt, \qquad F(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2\sin^2 t}}.$$

311. *Proposed by J. M. Gandhi, Lingraj College, Belgaum, India.*

Prove the following relation.

$$(2n+1)!!\ B_{2n} = (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & -\binom{3}{3} & \binom{3}{1} & \cdots & 0 \\ 1 & \binom{5}{5} & -\binom{5}{3} & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & (-1)^n\binom{2n+1}{2n+1} & (-1)\binom{2n+1}{2n-3} & \cdots & -\binom{2n+1}{3} \end{vmatrix}$$

where $(2n+1)!! = 1 \cdot 3 \cdot 5 \ldots (2n+1)$ and B_{2n} are the coefficients in the expansion of $\dfrac{\sinh n}{\sin n}$.

312. *Proposed by A. S. Gregory, Chicago, Illinois.*

Evaluate.

$$\sum_{m=4}^{n} \left\{ \sum_{i=0}^{[\frac{a-1}{2}]} \binom{a-1-i}{i} \sum_{j=0}^{[\frac{m-a}{2}]} \binom{m-a-j}{j} + \sum_{k=0}^{[\frac{a-2}{2}]} \binom{a-2-k}{k} \sum_{h=0}^{[\frac{m-a-1}{2}]} \binom{m-a-1-h}{h} \right\}$$

for $2 < a < m < n = 4, 5, 6, \ldots$.

SOLUTIONS

Errata: Problem 289, p. 102, Nov. 1956 should read, "Prove or disprove the following statement: The diagonal of a rectangular parallelopiped can be any odd integer other than one or five and the edges integers relatively prime to the diagonal." Problem 300, March 1957 should read, "Proposed by H. M. Gandhi."

LATE SOLUTIONS

278. *Richard K. Guy, University of Malaya, Singapore; Sister M. Stephanie, Georgian Court College, New Jersey.*

280. *D. A. Breault, Pittsburg, Pennsylvania; Richard K. Guy, University of Malaya, Singapore.*

279, 281, 282, 283, 284. *Richard K. Guy, University of Malaya, Singapore.*

286. *Miss V. R. Naik, Lingraj College, Belgaum, India.*

Bisecting Two Areas

285. [November 1956] *Proposed by Maj. H. S. Subba Rao, New Delhi. India.*

It is well known that given any two areas enclosed by closed curves in a plane there exists a straight line which bisects both the areas. Is it possible to construct such a line when the two areas are: a) a triangle and a parallelogram, b) two triangles?

Solution by Howard Eves, University of Maine. The straight lines bisecting the area of a triangle envelop a curvilinear triangle whose sides are arcs of determined hyperbolas having the sides of the triangle for asymptotes, and the straight lines bisecting the area of a parallelogram all pass through the center of the parallelogram. Therefore, the answer to part a) is yes, since tangents from a point to a determined hyperbola may be constructed with Euclidean tools, and the answer to part b) is no, since the common tangents to two determined hyperbolas cannot in general be constructed with Euclidean tools.

Also solved by Leon Bankoff, Los Angeles, California who pointed out two references. Mathematics Magazine, *Vol. 24, p 167, Jan. 1951, Problem 58 and Vol. 25, p 283, May 1952, Problem 90.*

A Trigonometric Integral

286. [November 1956] *Proposed by Joseph Andruskiw, Seton Hall University.*

Evaluate

$$I = \int_0^{\pi/2} x\left\{\frac{\sin x}{1+\cos^2 x} + \frac{\cos x}{1+\sin^2 x}\right\} dx$$

I. *Solution by A. K. Rajagopal, Belgaum, India.*

$$I = \int_0^{\pi/2} x\left\{\frac{\sin x}{1+\cos^2 x} + \frac{\cos x}{1+\sin^2 x}\right\} dx$$

$$= \int_0^{\pi/2} x\left\{d(\text{arc tan} \sin x) - d(\text{arc tan} \cos x)\right\}$$

$$= x \text{ arc tan sin } x \Big|_0^{\pi/2} - x \text{ arc tan cos } x \Big|_0^{\pi/2}$$

$$-\int_0^{\pi/2} \text{arc tan sin } x \, dx + \int_0^{\pi/2} \text{arc tan cos } x \, dx.$$

However the latter two integrals are equal and opposite in sign and

$$x \text{ arc tan sin } x \Big|_0^{\pi/2} = \pi/2 \text{ arc tan } 1 = \pi^2/8$$

and

$$x \text{ arc tan cos } x \Big|_0^{\pi/2} = 0. \text{ Hence } \quad I = \pi^2/8.$$

II. *Solution by Carman E. Miller, University of Saskatchewan.*

$$I = \int_0^{\pi/2} x \left\{ \frac{\sin x}{1 + \cos^2 x} + \frac{\cos x}{1 + \sin^2 x} \right\} dx.$$

Let J denote the integral obtained from I by deleting the factor x in the integrand. The form of the integrands and I and J suggests the substitution $x = \pi/2 - y$ in I and in the second part of J. This leads to the relations

$$I = \pi/2 \, J - I \quad \text{and} \quad J = 2 \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} \, dx$$

The latter integral is readily evaluated, say by the substitution $\cos x = t$. This yields $J = \pi/2$. Consequently $I = \pi^2/8$

Also solved by J. L. Alperin, Harvard College; Waleed A. Al-Salam, Duke University; John L. Brown Jr., Pennsylvania State University; Howard Eves, University of Maine; Sidney Glussman, New York, New York; Joseph D. E. Konhauser, State College, Pennsylvania; M. Morduchow, Polytechnic Institues of Brooklyn; Wahin Ng, San Francisco, California; Lawrence A. Ringenberg, Eastern Illinois State College; Robert E. Shafer, University of California Radiation Laboratory; Chih-yi Wang, University of Minnesota and the proposer.

A Cotangent Sum

287. [November 1956] *Proposed by Chih-yi Wang, University of Minnesota.*

Show that

$$\cot(\pi/11) - \cot(2\pi/11) + \cot(4\pi/11) - \cot(8\pi/11) + \cot(16\pi/11) = \sqrt{11}$$

I. *Solution by L. Carlitz, Duke University.*

Let $$S = \cot\frac{\pi}{11} - \cot\frac{2\pi}{11} + \cot\frac{4\pi}{11} - \cot\frac{8\pi}{11} + \cot\frac{16\pi}{11}.$$

If $\sigma = e^{2\pi i/11}$, then $$\cot\frac{k\pi}{11} = \frac{i\,\sigma^k + 1}{\sigma^k - 1},$$

$$S = i\left(\frac{\sigma+1}{\sigma-1} - \frac{\sigma^2+1}{\sigma^2-1} + \frac{\sigma^4+1}{\sigma^4-1} + \frac{\sigma^8+1}{\sigma^8-1} + \frac{\sigma^5+1}{\sigma^5-1}\right).$$

In the next place

$$(\sigma - 1)(\sigma + 2\sigma^2 + \dots + 10\sigma^{10}) = 11,$$

$$11\,\frac{\sigma+1}{\sigma-1} = -9\sigma - 7\sigma^2 - \dots + 9\sigma^{10},$$

which implies

$$11S = -11iG, \quad \text{where } G = \sum_{r=1}^{10}\left(\frac{r}{11}\right)\sigma^r,$$

where $\left(\frac{r}{11}\right)$ is the Legendre symbol. From the theory of Gauss sums (see for example Landau's *Vorlesungen über Zahlentheorie*, Vol. 1, p. 153)

$$G = i\sqrt{11}.$$

Therefore

$$S = \sqrt{11}.$$

II. *Solution by the proposer.* In the proof we use the following relations

(i) $\csc 2x = \cot x - \cot 2x$,

(ii) sum and difference formulas,

(iii) $\prod_{k=1}^{[n/2]} 2\sin\frac{k\pi}{n} = \sqrt{n}$, where n is an odd integer.

(see, for example, Nagell, *Introduction to Number Theory*, p. 173 or Whittaker and Watson, *Modern Analysis*, 4th ed., p. 240.)

$$S = \sum_{j=1}^{5} (-1)^{j-1} \cot(2^j\ \pi/11) = 1/2 \sum_{j=1}^{10} (-1)^{j-1} \cot(2^j\ \pi/11)$$

$$= 1/2 \sum_{k=1}^{5} \csc(2^{2k-1}\ \pi/11) = 1/2\,[\csc(\ \pi/11) + \csc(2\ \pi/11)$$

$$+ \csc(3\ \pi/11) - \csc(4\ \pi/11) + \csc(5\ \pi/11)].$$

An easy manipulation yields

$$S \prod_{k=1}^{5} \sin(k\ \pi/11) = 1/16\ [5 + \cos(\ \pi/11) - \cos(2\ \pi/11)$$

$$+ \cos(3\ \pi/11) - \cos(4\ \pi/11) + \cos(5\ \pi/11)] = \frac{11}{32}$$

for let $P = \cos(\ \pi/11) - \cos(2\ \pi/11) + \cos(3\ \pi/11) - \cos(4\ \pi/11)$

$+ \cos(5\ \pi/11)$ then $P \cos(\ \pi/22) = 1/2 \cos(11\ \pi/22)$.
Hence the result.

Also solved by A. K. Rajagopal, Lingraj College, Belgaum, India, who pointed out the relation of this problem to one appearing in Bromwich's "An Introduction to the Theory of Infinite Series", 1908, Macmillan and Co., London, Chapter X.

Fermat's Spiral

288. [November 1956] *Proposed by Norman Anning, Alhambra, California.*

A particle moves on a Fermat Spiral, $r^2 = a^2\theta$, under a central force directed to the origin. Show that the law of force involves the seventh power of the radius in a denominator.

I. *Solution by J. M. C. Hamilton, Los Angeles City College.*

The particle has kinetic energy $T = 1/2\, m(\dot{r}^2 + r^2\dot{\theta}^2)$. Since $\dot{r} = \dot{\theta}\,\dfrac{dr}{d\theta}$, $T = 1/2\, m\,[(\dfrac{dr}{d\theta})^2 + r^2]\,\dot{\theta}^2$. The angular momentum $J = mr^2\dot{\theta}$ is constant, because of conservation of angular momentum, hence we have

$T = \dfrac{J^2}{2mr^4}[(\dfrac{dr}{d\theta})^2 + r^2]$. Since the path of motion is $r^2 = a^2\theta$,

$\dfrac{dr}{d\theta} = \dfrac{a^2}{2r}$ and $T = \dfrac{J^2}{8m}\left(\dfrac{a^2 + 4r^4}{r^6}\right)$. The potential energy of the parti-

cle is $V = -\int_{-\infty}^{r} f(r)dr$ where $f(r)$ is the central force. By the principle of conservation of energy, $T + V = W$, a constant. Thus

$$\int_{-\infty}^{r} f(r)dr = -\frac{J^2}{4m}\left(\frac{a^4 + 4r^4}{r^7}\right) - W.$$

Differentiation with respect to r yeilds $f(r) = -\dfrac{J^2}{4m}\left(\dfrac{3a^4 + 4r^4}{r^7}\right)$ which establishes the desired result.

II. *Solution by Louis G. Vargo, Ramo-Wooldridge Corp., Los Angeles, California.* Using the areal velocity theorem, $r^2\dot{\theta} = h$ (a constant) so that the radial acceleration may be written in terms of $\mu = 1/r$ and its derivatives with respect to θ.

$$f = -mh^2\mu^2(\mu'' + \mu).$$

Evaluating μ'' and eliminating θ by means of the orbit equation, one obtains

$$f = -mh^2\mu^3(3a^4\mu^4 + 4)/4 = -mh^2(3a^4 + 4r^4)/4r^7.$$

Also solved by Sidney Glusman, New York, New York; J. M. C. Hamilton, Los Angeles City College (Second Solution)- Joseph D. E. Konhauser, State College, Pennsylvania; M. Morduchow, Polytechnic Institute of Brooklyn; A.K. Rajagopal, Lingraj College, Belgaum, India; Alan Wayne, Williamsburgh Vocational High School, Brooklyn, New York; Chih-yi Wang, University of Minnesota and the proposer.

Incircle Construction

290. [November 1956] *Proposed by Leon Bankoff, Los Angeles, California.*

The hypotenuse AC of a right triangle ABC is divided into unequal segments x and y by the point of contact of the incircle. Given segments x and y: **a)** construct the triangle ABC and its incircle, **b)** without using the fourth proportional, upon x as a base, construct a rectangle equivalent to triangle ABC.

I. *Solution by Dermott A. Breault, Pittsburgh, Pennsylvania.* **Part a.** Given x and y, and noting that the tangents to a circle from a point outside it are equal in length, we see that there exists a length u, such that

$$(x + y)^2 = (x + u)^2 + (y + u)^2 \tag{1}$$

Solving (1), we get:

$$u = \frac{-(x + y) + \sqrt{x^2 + 6xy + y^2}}{2} \tag{2}$$

(The (+) sign was chosen in (2) because u must be positive.)

Clearly, u is constructible, since all of the terms on the r.h.s. are constructible, and the required triangle is the one with sides $x + y$, $x + u$, $y + u$. Denoting the points of tangency of the incircle by a, b, and c, we see that the perpendicular bisectors of ac and bc meet at a point O, which is the center of the circle. Take the radius to be Oa, and the circle can easily be constructed.

Part b. The triangle has area $(1/2)(x + u)(y + u)$, which when multiplied out and divided by x gives:

$$1/2\left(y + u + \frac{uy}{x} + \frac{u^2}{x}\right). \tag{3}$$

On x as a base, construct a perpendicular to BC through B, and lay off on it a length equal to (3). We now have a rectangle whose dimensions are (3) by x, and its area is obviously equal to that of the triangle ABC.

II. *Solution by Sister M. Stephanie, Georgian Court College, New Jersey.* a. Erect perpendicular bisector of AC at O and perpendicular to AC at M, the point of contact of the incircle. With O as center and OC as radius, draw a circle. B lies (for definiteness) on upper semicircle. The perpendicular bisector cuts lower semicircle at K. With K as center and KA as radius, draw a circle. This cuts perpendicular erected to BC at M and I, the incenter. The incircle can now be constructed with I as center and IM as radius. With A as center and x as radius the point of tangency is found for side AB and with C as center and y as radius the point of tangency is found for side BC. Triangle ABC can now be drawn.

Proof: Consider line $BIKI'$ where I is incenter and I' is excenter with respect to vertex B. "The incenter of a triangle and the excenter relative to a given vertex are the extremeities of a diameter of a circle passing through the other two vertices of the triangle." Altshiller-Court, *College Geometry*, number 102. (Center of this circle is point common to II' and the perpendicular bisector of AC, for II' being the internal bisector of the angle B must pass through the midpoint K of the arc AC subtended by the side AC on the circumcircle of ABC):

b. Upon x as a base, construct a rectangle of altitude y. Proof follows from, "The area of a right triangle is equal to the product of the two segments into which the hypotenuse is divided by its point of contact with the incircle." Altshiller-Court, p. 81, problem 43.

Also solved by Howard Eves, University of Maine; L. A. Ringenberg Eastern Illinois State College; Chih-yi Wang, University of Minnesota and the proposer.

A Quotient of Means

291. [November 1956] *Proposed by M.S. Klamkin, Polytechnic Institute of Brooklyn.*

Determine the minimum of
$$\frac{\sum_{r=1}^{s} x_r^s}{\prod_{r=1}^{s} x_r} \quad \text{where } x_r > 0$$

Solution by Chih-yi Wang, University of Minnesota. Since the arithmetic mean is never less than the geometric mean of any positive terms we have

$$s\left\{\sum_{r=1}^{s} x_r^s/s\right\} \geq s \prod_{r=1}^{s} x_r$$

whence the required minimum value is s, which is attained if all x_r's are equal.

Also solved by Sidney Glusman Institute for Mathematical Sciences, New York University; Robert E. Shafer, University of California Radiation Laboratory and the proposer.

QUICKIES

From time to time this department will publish problems which ch may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant t solution and the source, if known.

Q 197. Find $f(x,y)$ such that the family $f(x,y) = c$ is orthogonal to the family $f(x,y) = k$. [*Submited by M. S. Klamkin*]

Q 198. Find the length of arc of the curve $P = e^{-\theta}$, $0 \leq \theta < \infty$, without integration. [*Submitted by Robert E. Shafer.*]

Q 199. Given a triangle ABC. Is there a point x not in the plane of ABC such that AX, BX and CX are mutually perpendicular? [*Submitted by B. K. Gold and J. M. Howell*]

Q 200. N is a random non negative integer. What is the probability that the last digit of the sum of the first N integers is a 7? [*Submitted by Geoffrey Matthews*]

Q 201. Determine a one parameter solution of

$$\frac{y''}{1 + (y')^2} = \sinh(y - x) - 1 \text{ . [Submitted by M.S. Klamkin]}$$

Q 202. Prove that if an isosceles right triangle is formed so that the equal sides are equal to the hypotenuse of a primitive Pythagorean triangle, there exists another right triangle with integral sides which has the same hypotenuse as the isosceles right triangle. [*Submitted by J. M. Howell*]

Q 203. Solve $\frac{dy}{dx} = x^2 + \sqrt{x^4 - 2xy}$ [*Submitted by N.S. Klambin*]

ANSWERS

A. 197 Such a family can be obtained by solving any differential equation of the form

$$\frac{dy}{dx} = [G(x,y) + G(y,x)][H(x) - H(y)]$$

A. 198. Let L represent the length of the curve. If we started instead at θ and found its arc length, we have two similar figures, and applying Cavalieri's Theorem we find the arc length of the second curve to be $Le^{-\theta}$. The chord length is found to be

$$\sqrt{(1 - e^{-\theta}\cos\theta)^2 + (e^{-\theta}\sin\theta)^2} = \sqrt{1 + e^{-2\theta} - 2e^{-\theta}\cos\theta}$$

and the sum of all chord lengths $(\bar{L})$ extending into the center is found by the relation:

$$\bar{L} - \bar{L}e^{-\theta} = \sqrt{1 + e^{-2\theta} - 2e^{-\theta}\cos\theta} \text{ or } \bar{L} = \sqrt{\frac{1 + e^{-2\theta} - 2e^{-\theta}\cos\theta}{1 + e^{-2\theta} - 2e^{-\theta}}}$$

hence

$$l = \lim_{\theta \to 0} \bar{L} = \lim_{\theta \to 0} \sqrt{\frac{1 + e^{-2\theta} - 2e^{-\theta}\cos\theta}{1 + e^{-2\theta} - 2e^{-\theta}}} = \sqrt{2}$$

A 199. Such a point x exists if and only if ABC is an acute triangle. Consider a plane cutting the corner of a room!

A 200. The probability is zero as the last digits of $\sum_{i=0}^{n} i$ form a sequence 0,1,3,6,0,5,1,8,6,5,5,6,8,1,5,0,6,3,1,0,... which repeats itself at this stage.

A 201. By inspection a first integral is $y' = \sinh(y - x)$. Let $z = y - x$, then $z' + 1 = \sinh z$ which is integrable into a one parameter solution.

A 202. If $a^2 + b^2 = c^2$ is satisfied with integers, then $(a - b)^2 + (a + b)^2 = 2c^2$ is also satisfied with integers.

A 203. To solve let $y = x^3 z$, then $x^3\frac{dz}{dx} + 3x^2 z = x^2(1 + \sqrt{1 - 2z})$. The variables are now seperable.